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Abstract: We study the behavior of the electromagnetic field in a biological cell modelized by a medium surrounded by a thin layer and embedded in an ambient medium. We derive approximate transmission conditions in order to replace the membrane by these conditions on the boundary of the interior domain. Our approach is essentially geometric and based on a suitable change of variables in the thin layer. Few notions of differential calculus are given in order to obtain our asymptotic conditions in a simple way. This paper extends to time-harmonic Maxwell equations the previous works presented in [27, 29, 28, 4]. Asymptotic transmission conditions at any order are given in Appendix 1.

Key-words: Asymptotic analysis, Maxwell's equations, differential calculus

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Conditions de transmission approchées pour les équations de Maxwell en régime harmonique dans un milieu à couche mince

Résumé : Nous étudions le comportement asymptotique du champ électromagnétique dans une cellule biologique plongée dans un milieu ambiant. La cellule est composée d'un cytoplasme entouré d'une fine membrane. Nous obtenons des conditions de transmission sur le bord du cytoplasme équivalentes à la couche mince. Notre approche est essentiellement géométrique et basée sur un changement de variables adéquat dans la couche mince. Quelques notions de calcul différentiel sont rappelées afin d'obtenir directement notre développement asymptotique. Par ailleurs des estimations d'erreur sont démontrées. En appendice, nous présentons le développement asymptotique à tout ordre.

Mots-clés : Analyse Asymptotique, Equations de Maxwell, Calcul différentiel

Contents

1 Introduction and motivations

The electromagnetic modelization of biological cells has become extremely important since several years, in particular in the biomedical research area. In the simple models [14, 16], the biological cell is a domain with thin layer composed of a conducting cytoplasm surrounded by a thin insulating membrane. When exposed to an electric field, a potential difference is induced across the cell membrane. This transmembrane potential (TMP) may be of sufficient magnitude to be biologically significant. In particular, if it overcomes a threshold value, complex phenomenons such as electropermeabilization or electroporation may occur [34, 33, 22, 21]. The electrostatic pressure becomes too high that the thin membrane is locally destructured: some exterior molecules might be internalized inside the cell. These process hold great promises particularly in oncology and gene therapy, to deliver drug molecules in cancer treatment. This is the reason why an accurate knowledge of the distribution of the electromagnetic field in the biological cell is necessary.

Several papers in the bioelectromagnetic research area deal with numerical electromagnetic modelizations of biological cells [23, 32, 30]. Actually the main difficulties of the numerical computations lie in the thinness of the membrane (the relative thickness of the membrane is of order 1/1000 compared with the size of the cell) and in the high contrast between the electromagnetic parameters of the cytoplasm and the membrane. We present here an asymptotic method to replace the thin membrane by appropriate transmission conditions on the boundary of the cytoplasm.

In previous papers [27, 29, 28, 4], an asymptotic analysis is proposed to compute the electric potential in domains with thin layer, using the electroquasistatic approximation[1]. However it is not clear that the magnetic effects of the field may be neglected. This is the reason why we present in this paper an asymptotic analysis for time-harmonic Maxwell equations in a domain with thin layer.

Our analysis is closed to those performed in [27, 29, 28]. Roughly speaking, it is based on a suitable change of variables in the membrane in order to write the explicit dependence of the studied differential operator in terms of the small parameter (the thinness of the membrane). Since we consider Maxwell equations in time-harmonic regime, in accordance with Flanders [15], Warnick *et al.* [35, 36] and Lassas *et al.* [17, 18], we choose the differential calculus formalism to perform our change of variables in a simple way. Basic notions of differential forms with explicit formulae are recalled in Appendix 2.

Throughout this paper, we consider a material composed of an interior domain surrounded by a thin membrane. This material, representing a biological cell, is embedded in an ambient medium submitted to an electric current density. We study the asymptotic behavior of the electromagnetic field in the three domains (the ambient medium, the thin layer and the cytoplasm) for the thickness of the membrane tending to zero. We derive appropriate transmission conditions on the boundary of the cytoplasm in order to remove the thin layer from the problem. Actually, the influence of the membrane is approached by these transmission conditions. To justify our asymptotic expansion, we provide

[1]The electroquasistatic approximation consists in considering that the electric field comes from a potential: $E = -\nabla V$. In this approximation the curl part of the electric field vanishes and the magnetic field is neglected.

piecewise estimates of the error between the exact solution and the approximate solution.

The paper is structured as follows. In Section 2, we present the studied problem in the differential calculus formalism and the main result of the paper. Section 3 is devoted to the geometry: we perform our change of variables and we write the problem in the so-called local coordinates. In Section 4 we derive formally our asymptotic expansion, which is rigorously proved in Section 5. In Appendix 1, we give recurrence formulae to obtain asymptotics at any order. Appendix 2 is devoted to the basic notions of differential forms used in the paper.

2 Maxwell equations using differential forms

In the following we use the conventions of differential calculus formalism. Definitions and basic properties of the differential calculus are given in Appendix 2. We recall here the usual notations.

Notation 2.1. *Let k be an integer. For a compact, connected and oriented Riemanian manifold $(M, \mathfrak{g})$ of $\mathbb{R}^n$ we denote by $\Omega^k(M)$ the space of k–forms defined on M.*

- *The exterior product between two differential forms ω and η is denoted by $\omega \wedge \eta$.*
- *The inner product on $\Omega^k(M)$ is denoted by $\left\langle \cdot, \cdot \right\rangle_{\Omega^k}$.*
- *The Hodge star operator is denoted by $\star$.*
- *The interior product of a differential form ω with a smooth vector field Y is written $\mathsf{int}(Y)\omega$.*

Let us denote the exterior differential and codifferential operators respectively by d, δ. The Laplace-Beltrami operator Δ is defined by $\Delta = -\mathrm{d}\delta - \delta\mathrm{d}$.

$L^2\Omega^k(M)$ is the space of the square integrable $k-forms$ of M while for $s \in \mathbb{R}$, $H^s\Omega^k(M)$ is the usual Sobolev space of k-forms. Let $H\Omega^k(\mathrm{d}, M)$ and $H\Omega^k(\delta, M)$ denote

$$H\Omega^k(\mathrm{d}, M) = \left\{\omega \in L^2\Omega^k(M) : \quad \mathrm{d}\omega \in L^2\Omega^{k+1}(M)\right\}, \tag{1}$$

$$H\Omega^k(\delta, M) = \left\{\omega \in L^2\Omega^k(M) : \quad \delta\omega \in L^2\Omega^{k-1}(M)\right\}, \tag{2}$$

that are Banach spaces when associated with their respective norms

$$\|\omega\|_{H\Omega^k(\mathrm{d},M)} = \|\omega\|_{L^2\Omega^k(M)} + \|\mathrm{d}\omega\|_{L^2\Omega^{k+1}(M)},$$
$$\|\omega\|_{H\Omega^k(\delta,M)} = \|\omega\|_{L^2\Omega^k(M)} + \|\delta\omega\|_{L^2\Omega^{k-1}(M)}.$$

We also denote by $H\Omega^k(\mathrm{d}, \delta, M)$ the space $H\Omega^k(\mathrm{d}, M) \cap H\Omega^k(\delta, M)$ equipped with the norm

$$\|\omega\|_{H\Omega^k(\mathrm{d},\delta,M)} = \|\omega\|_{L^2\Omega^k(M)} + \|\mathrm{d}\omega\|_{L^2\Omega^{k+1}(M)} + \|\delta\omega\|_{L^2\Omega^{k-1}(M)}.$$

$H^s(M)$ and $L^2(M)$ denotes the respective spaces $H^s\Omega^0(M)$ and $L^2\Omega^0(M)$.

2.1 The considered problem

Let Γ be a compact oriented surface of $\mathbb{R}^3$ without boundary. Consider the smooth connected bounded domain $\mathcal{O}_\mathsf{c}$ with boundary Γ; $\mathcal{O}_\mathsf{c}$ is surrounded by a thin layer $\mathcal{O}_{\mathsf{m}}^{\varepsilon}$ with constant thickness ε. This material with thin layer is embedded in an ambient smooth connected domain $\mathcal{O}_{\mathsf{e}}^{\varepsilon}$ with compact oriented boundary. We denote by $\mathcal{O}$ the ε-independent domain defined by

$$\mathcal{O} = \mathcal{O}_{\mathsf{e}}^{\varepsilon} \cup \overline{\mathcal{O}_{\mathsf{m}}^{\varepsilon}} \cup \mathcal{O}_{\mathsf{c}}.$$

Moreover, we denote by Γ_ε the boundary of $\overline{\mathcal{O}_{\mathsf{c}} \cup \mathcal{O}_{\mathsf{m}}^{\varepsilon}}$ (see **Figure** 1). Let μ_{c}, μ_{m} and μ_{e} be three positive constants and let q_{e}, q_{c} and q_{m} be three complex numbers. Define the two piecewise functions μ and q on $\mathcal{O}$ by

$$\forall x \in \mathcal{O}, \quad \mu(x) = \begin{cases} \mu_{\mathsf{e}}, \text{ in } \mathcal{O}_{\mathsf{e}}^{\varepsilon}, \\ \mu_{\mathsf{m}}, \text{ in } \mathcal{O}_{\mathsf{m}}^{\varepsilon}, \\ \mu_{\mathsf{c}}, \text{ in } \mathcal{O}_{\mathsf{c}}, \end{cases} \quad \forall x \in \mathcal{O}, \quad q(x) = \begin{cases} q_{\mathsf{e}}, \text{ in } \mathcal{O}_{\mathsf{e}}^{\varepsilon}, \\ q_{\mathsf{m}}, \text{ in } \mathcal{O}_{\mathsf{m}}^{\varepsilon}, \\ q_{\mathsf{c}}, \text{ in } \mathcal{O}_{\mathsf{c}}. \end{cases}$$

The function μ is the dimensionaless permeability of $\mathcal{O}$ while the function q denotes its dimensionaless complex permittivity[2].

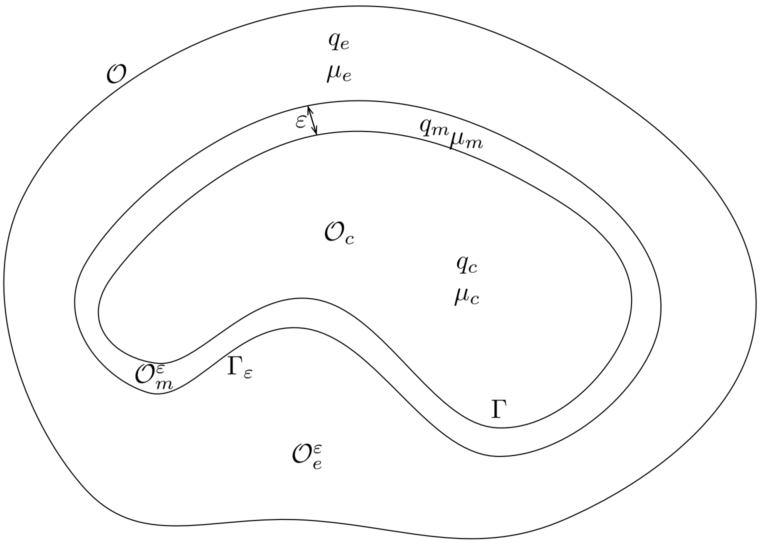

Figure 1: Geometry of the problem

Let $d_0 > 0$ such that for each point q of Γ, the normal lines of Γ passing through q, with center at q and length $2d_0$ are disjoints. In the following, we suppose that $\varepsilon \in (0, d_0)$. We denote by $\mathcal{O}_{\mathsf{e}}^{d_0}$ the set of points $\mathbf{x} \in \mathcal{O}_{\mathsf{e}}^{\varepsilon}$ at the distance greater than d_0 of Γ. We suppose that the ambient medium is submitted to a current density $\mathbb{J}$ compactly supported in $\mathcal{O}_{\mathsf{e}}^{d_0}$. All along the paper the following hypothesis holds.

Hypothesis 2.2. (i) *There exists $c_1, c_2 > 0$ such that for all $x \in \mathcal{O}$,*

$$c_1 \leqslant -\Im(q(x)) \leqslant c_2, \quad 0 < \Re(q(x)) \leqslant c_2. \tag{3}$$

[2]Using the notations of the electrical engineeering community, $q = \omega^2 \left(\epsilon - \mathrm{i}\frac{\sigma}{\omega}\right)$, where ω is the frequency, ϵ the permittivity and σ the conductivity of the domain [3].

(ii) *The source current density* $\mathbb{J}$ *satisfies*

$$\mathrm{supp}(\mathbb{J}) \Subset \mathcal{O}_{\mathsf{e}}^{d_0}, \quad \mathbb{J} \in L^2\Omega^1(\mathcal{O}), \quad \delta\mathbb{J} = 0, \ \textit{in } \mathcal{O}.$$

Maxwell equations describe the behavior of the electromagnetic field in $\mathcal{O}$. Denote by $\mathbb{E}$ and $\mathbb{H}$ the 1–forms representing respectively the electric and the magnetic fields in $\mathcal{O}$ in time-harmonic regime. Denote by $N_{\partial\mathcal{O}}$ the outward normal vector field to $\partial\mathcal{O}$. In the following and accordingly Remark 2.2 of Appendix 2, the normal vector field and the corresponding normal 1–form are identified. Maxwell equations in time-harmonic regime write [17, 18, 35]

$$\mathrm{d}\mathbb{E} = \mathrm{i} \star (\mu\mathbb{H}), \quad \mathrm{d}\mathbb{H} = -\mathrm{i} \star (q\mathbb{E} + \mathbb{J}), \text{ in } \mathcal{O}, \tag{4a}$$

$$N_{\partial\mathcal{O}} \wedge \mathbb{E}|_{\partial\mathcal{O}} = 0, \text{ on } \partial\mathcal{O}. \tag{4b}$$

Using the idempotence of $\star$ in $\mathbb{R}^3$, we may infer the vector wave equation on $\mathbb{E}$

$$\star\,\mathrm{d}\left(\frac{1}{\mu} \star \mathrm{d}\mathbb{E}\right) - q\mathbb{E} = \mathbb{J}, \text{ in } \mathcal{O}, \quad N_{\partial\mathcal{O}} \wedge \mathbb{E}|_{\partial\mathcal{O}} = 0, \text{ on } \partial\mathcal{O}.$$

Since μ is a scalar function[3] of $\mathcal{O}$, we infer

$$\delta\left(\frac{1}{\mu}\mathrm{d}\mathbb{E}\right) - q\mathbb{E} = \mathbb{J}, \text{ in } \mathcal{O}, \quad N_{\partial\mathcal{O}} \wedge \mathbb{E}|_{\partial\mathcal{O}} = 0, \text{ on } \partial\mathcal{O}. \tag{5}$$

Remark 2.3. Denote $\mathbb{E}$ in Euclidean coordinates by $E_x\mathrm{d}x + E_y\mathrm{d}y + E_z\mathrm{d}z$ and similarly for $\mathbb{H}$ and $\mathbb{J}$. Problem (4) and problem (5) write now

$$\mathrm{curl}\,\mathbb{E} = \mathrm{i}\mu\mathbb{H}, \quad \mathrm{curl}\,\mathbb{H} = -\mathrm{i}\,(q\mathbb{E} + \mathbb{J}), \text{ in } \mathcal{O}, \quad N_{\partial\Omega} \times \mathbb{E}|_{\partial\Omega} = O, \text{ on } \partial\mathcal{O},$$

and

$$\mathrm{curl}\left(\frac{1}{\mu}\,\mathrm{curl}\,\mathbb{E}\right) - q\mathbb{E} = \mathbb{J}, \text{ in } \mathcal{O}, \quad N_{\partial\Omega} \times \mathbb{E}|_{\partial\Omega} = 0, \text{ on } \partial\mathcal{O}.$$

△

2.2 Regularized variational formulation.

Our functional space $\mathbf{X}(q)$ is defined as

$$\mathbf{X}(q) = \left\{u \in H\Omega^1(\mathrm{d}, \mathcal{O}),\, \delta(qu) \in L^2(\mathcal{O}),\, N_{\partial\mathcal{O}} \wedge u|_{\partial\mathcal{O}} = 0\right\},$$

associated with its graph norm

$$\|u\|_{\mathbf{X}(q)} = \|u\|_{H\Omega^1(\mathrm{d},\mathcal{O})} + \|\delta(qu)\|_{L^2(\mathcal{O})}.$$

Define the sesquilinear form a_q in $\mathbf{X}(q)$ adapted to a regularized variational formulation of the problem (5) by

$$a_q(u,v) = \int_{\mathcal{O}} \left(\frac{1}{\mu}\left\langle \mathrm{d}u, \mathrm{d}\overline{v}\right\rangle_{\Omega^2} + \left\langle \delta(qu), \delta(\overline{qv})\right\rangle_{\Omega^0} - q\left\langle u, \overline{v}\right\rangle_{\Omega^1}\right) \mathrm{dvol}_{\mathcal{O}}\,.$$

Using inequalities (3), the following lemma holds.

[3] If μ is a tensor the previous equation (5) becomes $\delta\left(\star\mu^{-1} \star \mathrm{d}\mathbb{E}\right) - q\mathbb{E} = \mathbb{J}$.

Lemma 2.4. *There exists a constant $c_0 > 0$ and $\alpha \in \mathbb{C}$ such that for all $\varepsilon \in (0, d_0)$,*

$$\Re\big(\alpha a_q(u,u)\big) \geqslant c_0 \|u\|^2_{\mathbf{X}(q)}. \tag{6}$$

For all $\varepsilon \in (0, d_0)$, we consider the variational problem: find $\mathbb{E} \in \mathbf{X}(q)$ such that

$$\forall u \in \mathbf{X}(q), \quad a_q(\mathbb{E}, u) = \int_{\mathcal{O}} \langle \mathbb{J}, \overline{u} \rangle_{\Omega^1} \,\mathrm{dvol}_{\mathcal{O}} \,. \tag{7}$$

Using Hypothesis 2.2 the following theorem holds.

Theorem 2.5 (Equivalent problems)**.** *Let Hypothesis 2.2 hold.*
(i) *There is at most one solution $\mathbb{E} \in \mathbf{X}(q)$ to problem (7).*
(ii) *The solution $\mathbb{E}$ satisfies (5) in a weak sense*

$$\delta \mathrm{d}\mathbb{E} - \mu q \mathbb{E} = \mathbb{J}, \textit{in } \mathcal{O}^\varepsilon_{\mathsf{e}} \cup \mathcal{O}^\varepsilon_{\mathsf{m}} \cup \mathcal{O}_{\mathsf{c}}, \quad N_{\partial\mathcal{O}} \wedge \mathbb{E}|_{\partial\mathcal{O}} = 0,$$

with the divergence condition

$$\delta(q\mathbb{E}) = 0, \textit{in } \mathcal{O} \tag{8}$$

and the following equalities[4] *hold for $\mathscr{S} \in \{\Gamma, \Gamma_\varepsilon\}$*

$$[N_{\mathscr{S}} \wedge \mathbb{E}]_{\mathscr{S}} = 0, \quad \left[\frac{1}{\mu}\mathsf{int}(N_{\mathscr{S}})\mathrm{d}\mathbb{E}\right]_{\mathscr{S}} = 0, \quad [q\,\mathsf{int}(N_{\mathscr{S}})\mathbb{E}]_{\mathscr{S}} = 0. \tag{9}$$

(iii) *If $(\mathbb{E}, \mathbb{H}) \in \big(L^2\Omega^1(\mathcal{O})\big)^2$ is solution to problem (4) then $\mathbb{E} \in \mathbf{X}(q)$ satisfies (5). Conversely, if $\mathbb{E} \in \mathbf{X}(q)$ satisfies (5) then the couple of 1–forms $(\mathbb{E}, -(\mathrm{i}/\mu) \star \mathrm{d}\mathbb{E})$ belongs to $\big(L^2\Omega^1(\mathcal{O})\big)^2$ and satisfies problem (4).*

Proof. The proof is based on an idea of Costabel *et al.*.
(i) Accordingly estimate (6), a straightforward application of the well-known Lax-Milgram theorem leads to existence and uniqueness of the solution $\mathbb{E}$ to the regularized variational problem (7).
(ii) The proof is precisely worked out in full details in [5, 6] in a very slightly different configuration. We just give here the sketch of the proof. The first transmission conditions (9) easily come from Green formula of Proposition 2.8. The divergence conditions straightforwardly come from $\delta(q\mathbb{E}) = 0$, using (5) and $\delta^2 \equiv 0$. The other implication is obvious. We then prove that this solution satisfies $\delta(q\mathbb{E}) = 0$.
Denote by $H\Delta(\mathcal{O})$ the space of functions $\phi \in H^1_0(\mathcal{O})$ such that $\delta(q\mathrm{d}\phi)$ belongs to $L^2(\mathcal{O})$. Integrations by parts imply

$$\forall \phi \in H\Delta(\mathcal{O}), \quad a_q(\mathbb{E}, \mathrm{d}\phi) = \int_{\mathcal{O}} \langle \delta(q\mathbb{E}), \overline{\delta(q\mathrm{d}\phi) + \phi} \rangle_{\Omega^0} \,\mathrm{dvol}_{\mathcal{O}} \,.$$

Since $\Im(q) \leq -c_1 < 0$, the function $\delta(q\mathrm{d}\phi) + \phi$ runs through the whole $L^2(\mathcal{O})$ space for ϕ running through $H\Delta(\mathcal{O})$. Moreover, since $\delta(\mathbb{J})$ vanishes we have

$$\int_{\mathcal{O}} \langle \mathbb{J}, \overline{\mathrm{d}\phi} \rangle_{\Omega^1} \,\mathrm{dvol}_{\mathcal{O}} = 0,$$

[4]For an oriented surface $\mathscr{S}$ without boundary and for a differential form u defined in a neighborhood of $\mathscr{S}$ we denote by $[u]_{\mathscr{S}}$ the jump across $\mathscr{S}$. $N_{\mathscr{S}}$ denotes the outward normal to $\mathscr{S}$

from which we infer that $\delta(q\mathbb{E})$ identically vanishes in $L^2(\mathcal{O})$ according to (7). Therefore the solution $\mathbb{E}$ of problem (7) solves problem (5).
(iii) If $(\mathbb{E},\mathbb{H})$ solves problem (4) we straightforwadly infer (5), since $\star$ is idempotent and since μ is a scalar function. Conversely, defining $\mathbb{H}$ by

$$\mathbb{H} = -\frac{\mathrm{i}}{\mu} \star \mathrm{d}\mathbb{E},$$

we infer that $(\mathbb{E},\mathbb{H})$ solves problem (4). □

Denote by $\mathcal{O}_{\mathsf{e}}$ the domain $\mathcal{O}_{\mathsf{e}} = \mathcal{O} \setminus \overline{\mathcal{O}_{\mathsf{c}}}$. Define $\tilde{\mu}$ and $\tilde{q}$ by

$$\forall x \in \mathcal{O}, \quad \tilde{\mu}(x) = \begin{cases} \mu_{\mathsf{c}}, \text{ in } \mathcal{O}_{\mathsf{c}}, \\ \mu_{\mathsf{e}}, \text{ in } \mathcal{O}_{\mathsf{e}}, \end{cases} \quad \forall x \in \mathcal{O}, \quad \tilde{q}(x) = \begin{cases} q_{\mathsf{c}}, \text{ in } \mathcal{O}_{\mathsf{c}}, \\ q_{\mathsf{e}}, \text{ in } \mathcal{O}_{\mathsf{e}}. \end{cases}$$

Let $\mathbb{E}^0 \in \mathbf{X}(\tilde{q})$ be the "background" solution defined by

$$\forall u \in \mathbf{X}(\tilde{q}), \quad a_{\tilde{q}}(\mathbb{E}^0, u) = \int_{\mathcal{O}} \langle \mathbb{J}, \overline{u} \rangle_{\Omega^1} \, \mathrm{dvol}_{\mathcal{O}},$$

which means in a weak sense

$$\delta\left(\frac{1}{\tilde{\mu}} \mathrm{d}\mathbb{E}^0\right) - \tilde{q}\,\mathbb{E}^0 = \mathbb{J}, \text{ in } \mathcal{O}, \quad N_{\partial\mathcal{O}} \wedge \mathbb{E}^0|_{\partial\mathcal{O}} = 0. \tag{10}$$

We have the following regularity result.

Proposition 2.6. *Let Hypothesis 2.2 hold. Moreover let $s \geq 0$ and $\mathbb{J}$ belong to $H^s\Omega^1(\mathcal{O}_{\mathsf{e}}^{d_0})$. Then the 1-form $\mathbb{E}^0$ exists and is unique in $\mathbf{X}(\tilde{q})$. Moreover, denoting by $\mathbb{E}^{\mathsf{c},0}$ and $\widetilde{\mathbb{E}}^{\mathsf{e},0}$ its respective restrictions to $\mathcal{O}_{\mathsf{c}}$ and $\mathcal{O}_{\mathsf{e}}$, we have*

$$\widetilde{\mathbb{E}}^{\mathsf{e},0} \in H^{2+s}\Omega^1(\mathcal{O}_{\mathsf{e}}), \quad \mathbb{E}^{\mathsf{c},0} \in H^{2+s}\Omega^1(\mathcal{O}_{\mathsf{c}}).$$

Proof. The 1–form $\mathbb{E}^0$ satisfies (10). The proof of the existence and the uniqueness of $\mathbb{E}^0$ in $\mathbf{X}(\tilde{q})$ is very similar to those performed in Theorem 2.5 by replacing $\mathbf{X}(q)$ by $\mathbf{X}(\tilde{q})$ and a_q by $a_{\tilde{q}}$. Since $\delta\mathbb{J}$ vanishes, we infer $\delta(\tilde{q}\mathbb{E}^0) = 0$ and therefore $\mathbb{E}^0$ satisfies

$$-\Delta\mathbb{E}^0 - \tilde{\mu}\tilde{q}\,\mathbb{E}^0 = \mathbb{J}, \text{ in } \mathcal{O}_{\mathsf{e}} \cup \mathcal{O}_{\mathsf{c}}, \quad N_{\partial\mathcal{O}} \wedge \mathbb{E}^0|_{\partial\mathcal{O}} = 0,$$

with transmission conditions

$$\begin{aligned} &\left[N_\Gamma \wedge \mathrm{d}\mathbb{E}^0\right]_\Gamma = 0, \quad \left[\tilde{q}\,\mathsf{int}(N_\Gamma)\mathbb{E}^0\right]_\Gamma = 0, \\ &\left[\frac{1}{\tilde{\mu}}\mathsf{int}(N_\Gamma)\mathrm{d}\mathbb{E}^0\right]_\Gamma = 0, \quad \left[\delta(\tilde{q}\mathbb{E}^0)\right]_\Gamma = 0. \end{aligned}$$

The same calculations as performed in Proposition 2.1 of Costabel *et al.* [6] imply that the set of the above transmission and boundary conditions covers[5] the Laplacian in $\mathcal{O}_{\mathsf{c}}$ and in $\mathcal{O}_{\mathsf{e}}$, in the sense of Definition 1.5 at page 125 of Lions and Magenes [20]. Therefore we infer the piecewise elliptic regularity of $\mathbb{E}^0$, which ends the proof of the lemma. □

[5] Accordingly the appendix of the paper of Li and Vogelius [19] the regularity of $\mathbb{E}^0$ may also be obtained by using a reflection to reduce the problem to an elliptic system with complementing boundary conditions in the sense of Agmon *et al.*[1, 2].

The following estimates hold

Proposition 2.7. *Under Hypothesis 2.2, there exists* $C > 0$ *such that for all* $\varepsilon \in (0, d_0)$

$$\|\mathbb{E}\|_{\mathbf{X}(q)} \leqslant C, \tag{11}$$

$$\|\mathbb{E} - \mathbb{E}^0\|_{H\Omega^1(\mathrm{d},\mathcal{O})} \leqslant C\sqrt{\varepsilon}. \tag{12}$$

Proof. Using (6), estimates (11) are obvious since $\Im(q) \leqslant -c_1 < 0$. Prove now (12). We first mention that $\mathbb{E}^0$ belongs to $H^2\Omega^1(\varpi)$ for $\varpi \in \{\mathcal{O}_\mathsf{e}, \mathcal{O}_\mathsf{c}\}$, according to Proposition 2.6; hence $\mathbb{E}^0 \in L^\infty\Omega^1(\varpi)$ and $\mathrm{d}\mathbb{E}^0 \in L^\infty\Omega^2(\varpi)$. Denoting by $\mathbb{U} = \mathbb{E} - \mathbb{E}^0$ we infer

$$\int_{\mathcal{O}} \frac{1}{\mu} \langle \mathrm{d}\mathbb{U}, \overline{\mathrm{d}\mathbb{U}} \rangle_{\Omega^2} - q\langle \mathbb{U}, \overline{\mathbb{U}} \rangle_{\Omega^1} \,\mathrm{dvol}_{\mathcal{O}} = q_\mathsf{m} \int_{\mathcal{O}^\varepsilon_\mathsf{m}} \langle \mathbb{E}^0, \overline{\mathbb{U}} \rangle_{\Omega^1} \,\mathrm{dvol}_{\mathcal{O}^\varepsilon_\mathsf{m}}$$
$$- \frac{1}{\mu_\mathsf{m}} \int_{\mathcal{O}^\varepsilon_\mathsf{m}} \langle \mathrm{d}\mathbb{E}^0, \overline{\mathrm{d}\mathbb{U}} \rangle_{\Omega^2} \,\mathrm{dvol}_{\mathcal{O}^\varepsilon_\mathsf{m}}.$$

Therefore using (11) and using the assumption (3) on q, we infer

$$\|\mathrm{d}\mathbb{U}\|_{L^2\Omega^2(\mathcal{O})} + \|\mathbb{U}\|_{L^2\Omega^1(\mathcal{O})} \leqslant C\sqrt{\varepsilon}.$$

□

2.3 Main result

Consider the inclusion $\mathcal{J} : \Gamma \longrightarrow \mathcal{O}$, and $\mathcal{J}^*$ its pull-back $\mathcal{J}^* : \Omega^k(\mathcal{O}) \longrightarrow \Omega^k(\Gamma)$, $k \in \{0, 1, 2, 3\}$. Denote by d_Γ and δ_Γ the exterior differential and the codifferential operators defined on $\Omega^k(\Gamma)$. Define $\mathbb{S}$ and $\mathbb{T}$ by

$$\mathbb{S} = (q_\mathsf{m} - q_\mathsf{e})\, \mathcal{J}^*(\mathbb{E}^0)|_{\Gamma^+} + \left(\frac{1}{\mu_\mathsf{m}} - \frac{1}{\mu_\mathsf{e}}\right) \delta_\Gamma \mathrm{d}_\Gamma \left(\mathcal{J}^*(\mathbb{E}^0)\right)|_{\Gamma^+}, \tag{13}$$

$$\mathbb{T} = q_\mathsf{c} \left(\frac{1}{q_\mathsf{m}} - \frac{1}{q_\mathsf{e}}\right) \mathrm{d}\Big(\mathsf{int}(N_\Gamma)\mathbb{E}^0|_{\Gamma^-}\Big) + \frac{\mu_\mathsf{m} - \mu_\mathsf{e}}{\mu_\mathsf{c}}\, \mathsf{int}(N_\Gamma) \left(\mathrm{d}\mathbb{E}^0\right)\big|_{\Gamma^-}. \tag{14}$$

The explicit expressions of $\mathbb{S}$ and $\mathbb{T}$ in local coordinates are given in Section 5. Let $\mathbb{E}^1$ be the 1–forms defined by

$$\delta\mathrm{d}\mathbb{E}^1 - \tilde{\mu}\tilde{q}\mathbb{E}^1 = 0, \text{ in } \mathcal{O}_\mathsf{e} \cup \mathcal{O}_\mathsf{c}, \quad N_{\partial\mathcal{O}} \wedge \mathbb{E}^1|_{\partial\mathcal{O}} = 0,$$

with the following transmission conditions on Γ

$$\frac{1}{\mu_\mathsf{e}}\mathsf{int}(N_\Gamma)\mathrm{d}\mathbb{E}^1|_{\Gamma^+} - \frac{1}{\mu_\mathsf{c}}\mathsf{int}(N_\Gamma)\mathrm{d}\mathbb{E}^1|_{\Gamma^-} = \mathbb{S}, \tag{15}$$

$$N_\Gamma \wedge \mathbb{E}^1|_{\Gamma^+} - N_\Gamma \wedge \mathbb{E}^1|_{\Gamma^-} = N_\Gamma \wedge \mathbb{T}. \tag{16}$$

The aim of this paper is to prove the following theorem.

Theorem 2.8. *Under Hypothesis 2.2, if moreover the current density* $\mathbb{J}$ *belongs to* $H^3\Omega^1(\mathcal{O}^{d_0}_\mathsf{e})$*, there exists* $\varepsilon_0 > 0$ *and a constant* C*, independent on* ε *such that*

$$\forall\varepsilon \in (0, \varepsilon_0), \quad \|\mathbb{E} - (\mathbb{E}^0 + \varepsilon\mathbb{E}^1)\|_{H\Omega^1(\mathrm{d},\delta,\mathcal{O}_\mathsf{c})} \leqslant C\varepsilon^2,$$

and for any domain ϖ *compactly embedded in* $\mathcal{O}_\mathsf{e}$*, there exists* $\varepsilon_\varpi > 0$ *and a constant* $C_\varpi > 0$ *independent on* ε *such that*

$$\forall\varepsilon \in (0, \varepsilon_\varpi), \quad \|\mathbb{E} - (\mathbb{E}^0 + \varepsilon\mathbb{E}^1)\|_{H\Omega^1(\mathrm{d},\delta,\varpi)} \leqslant C_\varpi\varepsilon^2.$$

Remark 2.9. It is possible to give a precise behavior of $\mathbb{E}$ in a neighborhood of Γ by defining a 1–form in the thin membrane (see Theorem 5.3). △

In this paper we choose to deal with differential forms, in accordance with Flanders [15]. This point of view has the convenience of considering both electric and magnetic fields as 1–forms, *i.e.* they are physically similar in accordance with electrical engineering considerations [3]. It is also possible to derive our asymptotics by tensor calculus considerations, as used in linear elasticity of thin shell [7, 12, 13]. This approach is worked out in full details in the thesis [25] of the first author.

Remark 2.10. [The tensor calculus formulation] Since we are confident that our result might be very useful for bioelectromagnetic computations, and since the electrical engineering community may feel uncomfortable with the differential calculus formalism, we translate our result with the help of "usual" differential operators. Denote by ∇_Γ and $\nabla_\Gamma\cdot$ the respective gradient and divergence operators on Γ. Define Rot_Γ and rot_Γ by

$$\begin{aligned}
&\forall f \in C^\infty(\Gamma), \quad \mathrm{Rot}_\Gamma\, f = (\nabla_\Gamma f) \times N_\Gamma,\\
&\forall \mathbf{f} \in \left(C^\infty(\Gamma)\right)^3, \quad \mathrm{rot}_\Gamma\, \mathbf{f} = \nabla_\Gamma \cdot (\mathbf{f} \times N_\Gamma)\,.
\end{aligned}$$

Then $(\mathbb{E}^k)_{k=0,1}$ (seen as vector field) satisfy the following equations

$$\mathrm{curl}\,\mathrm{curl}\,\mathbb{E}^k - \tilde{\mu}\tilde{q}\mathbb{E}^k = \delta_0^k \mathbb{J}, \text{ in } \mathcal{O}_\mathsf{e} \cup \mathcal{O}_\mathsf{c}, \quad N_{\partial\mathcal{O}} \times \mathbb{E}^k|_{\partial\mathcal{O}} = 0,$$

with the following transmission conditions on Γ

$$N_\Gamma \times \mathbb{E}^0|_{\Gamma+} = N_\Gamma \times \mathbb{E}^0|_{\Gamma-},\ \frac{1}{\mu_\mathsf{e}}\left(N_\Gamma \times \mathrm{curl}\,\mathbb{E}^0\right)|_{\Gamma+} = \frac{1}{\mu_\mathsf{c}}\left(N_\Gamma \times \mathrm{curl}\,\mathbb{E}^0\right)|_{\Gamma-}, \tag{17}$$

$$\begin{aligned}
N_\Gamma \times \mathbb{E}^1|_{\Gamma+} \times N_\Gamma = N_\Gamma \times \mathbb{E}^1|_{\Gamma-} \times N_\Gamma &+ q_\mathsf{c}\left(\frac{1}{q_\mathsf{m}} - \frac{1}{q_\mathsf{e}}\right)\nabla_\Gamma\left(\mathbb{E}^0|_{\Gamma-}\cdot N_\Gamma\right)\\
&+ \frac{\mu_\mathsf{m}-\mu_\mathsf{e}}{\mu_\mathsf{c}}\left(\mathrm{curl}\,\mathbb{E}^0 \times N_\Gamma\right)|_{\Gamma-},
\end{aligned} \tag{18}$$

$$\begin{aligned}
\frac{1}{\mu_\mathsf{e}}\left(\mathrm{curl}\,\mathbb{E}^1 \times N_\Gamma\right)|_{\Gamma+} = \frac{1}{\mu_\mathsf{c}}\left(\mathrm{curl}\,\mathbb{E}^1 \times N_\Gamma\right)|_{\Gamma-} &+ (q_\mathsf{m} - q_\mathsf{e})\left(\mathrm{N}_\Gamma \times \mathbb{E}^0 \times N_\Gamma\right)|_{\Gamma+}\\
&+ \left(\frac{1}{\mu_\mathsf{m}} - \frac{1}{\mu_\mathsf{e}}\right)\mathrm{Rot}_\Gamma\,\mathrm{rot}_\Gamma\left(\mathrm{N}_\Gamma \times \mathbb{E}^0 \times N_\Gamma\right)|_{\Gamma+}.
\end{aligned}$$

△

Remark 2.11. [The impedance boundary condition of Engquist–Nédélec [11]] Let $\mathbb{J}$ be supported in $\mathcal{O}_\mathsf{c}$ (and divergence free) and suppose that $\mathcal{O}_\mathsf{e}^\varepsilon$ is a perfectly conducting domain. Therefore $q_\mathsf{e} = +\infty$ and $\mu_\mathsf{e} = 0$. An homogeneous Dirichlet condition is imposed on Γ_ε

$$N_{\Gamma_\varepsilon} \times \mathbb{E}|_{\Gamma_\varepsilon} = 0.$$

We are now in the same configuration as the problem studied by Engquist and Nédélec [11] at page 18. According to (17)–(18), writing the condition satisfied by $\mathbb{E}^0 + \varepsilon\mathbb{E}^1$ and neglecting the terms in ε^2, we infer the following boundary condition for the first-order approximate field $\mathbb{E}_a$

$$N_\Gamma \times \mathbb{E}_a|_{\Gamma^-} \times N_\Gamma = -\varepsilon \left(\frac{q_\mathsf{c}}{q_\mathsf{m}} \nabla_\Gamma \left(\mathbb{E}_a|_{\Gamma^-} \cdot N_\Gamma \right) + \frac{\mu_\mathsf{m}}{\mu_\mathsf{c}} \left(\operatorname{curl} \mathbb{E}_a \times N_\Gamma \right)|_{\Gamma^-} \right).$$

Recall that according to Maxwell equations, $\operatorname{curl}\mathbb{E} = \mathrm{i}\mu_\mathsf{c}\mathbb{H}$ and $\operatorname{curl}\mathbb{H} = -\mathrm{i}q_\mathsf{c}\mathbb{E}$. Therefore $q_\mathsf{c}\mathbb{E}\cdot N_\Gamma = \mathrm{i}\operatorname{curl}\mathbb{H}\cdot N_\Gamma$. According to the definition of $\nabla_\Gamma\cdot$ (see for example equation (2.22) page 5 of [11]), we infer[6]

$$\nabla_\Gamma \cdot (\mathbb{H} \times N_\Gamma) = \operatorname{curl}\mathbb{H}\cdot N_\Gamma = -\mathrm{i}q_\mathsf{c}\mathbb{E}\cdot N_\Gamma, \tag{19}$$

and the impedance boundary condition follows

$$N_\Gamma \times \mathbb{E}_a|_{\Gamma^-} \times N_\Gamma = -\mathrm{i}\varepsilon \left(\frac{1}{q_\mathsf{m}} \nabla_\Gamma \left(\nabla_\Gamma \cdot (\mathbb{H}_a \times N_\Gamma) \right) + \mu_\mathsf{m} \left(\mathbb{H}_a \times N_\Gamma \right)|_{\Gamma^-} \right).$$

Observe that this is the impedance boundary condition of given in [11] page 19, since they took the normal interior to their domain Ω_∞, hence $n = -N_\Gamma$. $\triangle$

We point out few arguments to enlight the convenience of differential calculus formalism.

(i) **Anisotropy.** For sake of simplicity, we deal here with isotropic materials, however the anisotropic case may be interesting for applications. In this case, μ and q are matrices and the vector wave equation becomes

$$\delta\left(\left(\star\mu^{-1}\star\right)\mathrm{d}\mathbb{E}\right) - q\mathbb{E} = \mathbb{J}, \text{ in } \mathcal{O} \quad N_{\partial\mathcal{O}} \wedge \mathbb{E}|_{\partial\mathcal{O}} = 0, \text{ on } \partial\mathcal{O},$$

and the following transmission conditions hold on $\mathscr{S} \in \{\Gamma, \Gamma_\varepsilon\}$

$$\left[\mathsf{int}(N_\mathscr{S})\left(\star\mu^{-1}\star\mathrm{d}\mathbb{E}\right)\right]_\mathscr{S} = 0, \quad \left[N_\mathscr{S}\wedge\mathbb{E}\right]_\mathscr{S} = 0.$$

To obtain the approximate transmission conditions equivalent to the thin layer, we just have to write the tensor $\star\mu^{-1}\star$ in local coordinates, with the help of the explicit formulae given in Appendix 2. The calculations are more tedious but we are confident that the reader has all the tools to perform the analysis.

(ii) **Non-constant thickness.** We consider here a thin layer with constant thickness. As mentionned in Section 1 a high electric field may occur a dramatically local decreasing of the thickness of the membrane, possibly leading to the apparition of pores. Hence the thickness of the membrane is no more constant with respect to the tangential variable. As performed in [28], the change of variables would lead to additional terms in the transmission conditions. These terms would come from the fact that the coefficients g_{i3} of the matrix (g_{ij}) given by (20) do not vanish. The derivation of the asymptotics would be more tedious but, once again, we are confident that all the tools are given in the present paper to perform the calculation.

In the case of a rough thin layer, the present analysis may not be applied. We have to introduce appropriate correctors as performed in [4].

[6] Using differential forms and since $\mathrm{d}N = 0$ equality (2.05) implies

$\mathsf{int}(N_\Gamma)\mathbb{E}^0|_{\Gamma^-} = -\frac{1}{\mathrm{i}q_\mathsf{c}}\mathsf{int}(N_\Gamma)\delta\left(\star\mathbb{H}^0\right) = -\frac{1}{\mathrm{i}q_\mathsf{c}}\delta_\Gamma\left(\mathsf{int}(N)\star\mathbb{H}^0|_\Gamma\right)$, which is exactly equality (19).

(iii) **Link with Helmholtz equation.** Observe that equations (5) are well-defined if $\mathbb{E}$ and $\mathbb{J}$ are functions, since the operators d and δ are defined for k–forms and the exterior product between a 1–form and a function is also well-defined. Moreover, since δ acting on functions is zero, the operator $-\delta\mathrm{d}$ coincides with Laplace-Beltrami operator Δ. In addition, the above differential forms $\mathbb{S}$ and $\mathbb{T}$ are well-defined even if $\mathbb{E}^0$ is a function, and in this case we have

$$\mathbb{S} = (q_\mathsf{m} - q_\mathsf{e})\,\mathbb{E}^0|_{\Gamma+} + \left(\frac{1}{\mu_\mathsf{m}} - \frac{1}{\mu_\mathsf{e}}\right)\delta_\Gamma \mathrm{d}_\Gamma\left(\mathbb{E}^0\right)|_{\Gamma+},$$
$$\mathbb{T} = \frac{\mu_\mathsf{m} - \mu_\mathsf{e}}{\mu_\mathsf{c}}\,\mathsf{int}(N_\Gamma)\left(\mathrm{d}\mathbb{E}^0\right)\big|_{\Gamma-},$$

since the interior product $\mathsf{int}(N_\Gamma)$ acting on functions is zero. Writing formally our asymptotic transmission conditions for functions in tensor calculus formalism, we infer that the function u solution to

$$-\nabla\cdot\left(\frac{1}{\mu}\nabla u\right) - qu = j, \text{ in } \mathcal{O}, \quad u|_{\partial\mathcal{O}} = 0,$$

is approached by $u^0 + \varepsilon u^1$ where $(u^k)_{k=0,1}$ satisfy

$$-\Delta u^k - \tilde{\mu}\tilde{q}u^k = \delta_0^k j, \quad \text{in } \mathcal{O}_\mathsf{c}\cup\mathcal{O}_\mathsf{e}, \quad u^k|_{\partial\mathcal{O}} = 0,$$

with the following transmission conditions

$$[u^0]_\Gamma = 0, \quad \left[\frac{1}{\tilde{\mu}}\partial_n u^0\right]_\Gamma = 0, \quad u^1|_{\Gamma+} - u^1|_{\Gamma-} = \frac{\mu_\mathsf{m} - \mu_\mathsf{e}}{\mu_\mathsf{c}}\partial_n u^0|_{\Gamma-},$$
$$\frac{1}{\mu_\mathsf{e}}\partial_n u^1|_{\Gamma+} - \frac{1}{\mu_\mathsf{c}}\partial_n u^1|_{\Gamma-} = (q_\mathsf{m} - q_\mathsf{e})\,u^0|_{\Gamma+} - \left(\frac{1}{\mu_\mathsf{m}} - \frac{1}{\mu_\mathsf{e}}\right)\Delta_\Gamma u^0|_{\Gamma+}.$$

This asymptotic expansion is rigorously proved in [26] (see equations (4) page 4 of [26]). Therefore the differential calculus provide a link between the results for Helmholtz and Maxwell equations.

3 Geometry

Let $\mathscr{V}_\Gamma$ be the tubular open neighborhood of Γ composed by the points at the distance d_0 of Γ. In the following, it will be convenient to write the involved differential form $\mathbb{E}$ in local coordinates in the tubular neighborhood $\mathscr{V}_\Gamma$ of Γ. We denote by $\mathscr{V}_\mathsf{e}^\varepsilon$ and $\mathscr{V}_\mathsf{c}$ the respective intersections $\mathscr{V}_\Gamma \cap \mathcal{O}_\mathsf{e}^\varepsilon$ and $\mathscr{V}_\Gamma \cap \mathcal{O}_\mathsf{c}$.

3.1 Parameterization of Γ.

Let $\mathsf{x}_\mathsf{T} = (x_1, x_2)$ be a system of local coordinates on $\Gamma = \{\psi(\mathsf{x}_\mathsf{T})\}$. By abuse of notations, we denote by $\mathsf{x}_\mathsf{T} \in \Gamma$ the point of Γ equal to $\psi(\mathsf{x}_\mathsf{T})$. In the (x_1, x_2)–coordinates, we denote by N_Γ the outward vector normal to Γ defined by

$$N_\Gamma = \frac{\partial_1\psi\wedge\partial_2\psi}{\|\partial_1\psi\wedge\partial_2\psi\|},$$

and we define by Φ the following map

$$\forall(\mathsf{x}_\mathsf{T}, x_3) \in \Gamma\times\mathbb{R}, \quad \Phi(\mathsf{x}_\mathsf{T}, x_3) = \psi(\mathsf{x}_\mathsf{T}) + x_3 N_\Gamma(\mathsf{x}_\mathsf{T}).$$

Notation 3.1. *In the following ∂_j stands for ∂_{x_j} for $j = 1, 2, 3$. Moreover we use the summation indices convention $a_i b_i = \sum_{i=1,2,3} a_i b_i$. Observe that according to our change of variables, $\mathsf{x_T}$ denotes the tangential variables and x_3 is the normal direction. To accentuate the difference between $\mathsf{x_T}$ and x_3, the Greek letters α and β (and possibly γ, ι, κ and λ) denote the indices in $\{1, 2\}$, while the letters i, j, k denote the indices in $\{1, 2, 3\}$. Eventually it is convenient to introduce the Levi-Cività symbol ϵ_{ijk} defined by*

$$\epsilon_{ijk} = \begin{cases} +1, \text{ if } \{i,j,k\} \text{ is an even permutation of } \{1,2,3\}, \\ -1, \text{ if } \{i,j,k\} \text{ is an odd permutation of } \{1,2,3\}, \\ 0, \text{ if any two labels are the same.} \end{cases}$$

According to the definition of d_0, the tubular neighborhood $\mathscr{V}_\Gamma$ of Γ may be parameterized by

$$\mathscr{V}_\Gamma = \{\Phi(\mathsf{x_T}, x_3), \quad (\mathsf{x_T}, x_3) \in \Gamma \times (-d_0, d_0)\}.$$

The system of coordinates $(\mathsf{x_T}, x_3)$ is the so-called local coordinates of $\mathscr{V}_\Gamma$. The Euclidean metric of $\mathscr{V}_\Gamma$ written in $(\mathsf{x_T}, x_3)$–coordinates is given by the following matrix $(g_{ij})_{i,j=1,2,3}$

$$(g_{ij})_{i,j=1,2,3} = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{12} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{20}$$

where the coefficient $g_{\alpha\beta}$ equals $g_{\alpha\beta} = \langle \partial_\alpha \Phi, \partial_\beta \Phi \rangle$. Here $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product of $\mathbb{R}^3$. Denote by (g^{ij}) the inverse matrix of (g_{ij}) , and by g the determinant of (g_{ij}). The coefficients $g_{\alpha\beta}$ might be written with the help of the coefficients of the first, the second and of the third fundamental forms of Γ in the basis $(\partial_1 \psi, \partial_2 \psi)$ (see Do Carmo [8])

$$g_{\alpha\beta}(\mathsf{x_T}, x_3) = g^0_{\alpha\beta}(\mathsf{x_T}) - 2x_3 b_{\alpha\beta}(\mathsf{x_T}) + x_3^2 c_{\alpha\beta}(\mathsf{x_T}).$$

The mean curvature $\mathscr{H}$ of Γ equals

$$\mathscr{H} = -\frac{1}{2} \left. \frac{\partial_3 \left(\sqrt{g}\right)}{\sqrt{g}} \right|_{x_3=0}. \tag{21}$$

3.2 The transmission conditions in local coordinates

In the $(\mathsf{x_T}, x_3)$–coordinates, write $\mathbb{E} = E_i \mathrm{d}x^i$. N_Γ is the outward normal field of Γ, which is identified to the 1–form $\mathrm{d}x^3$ in accordance with Remark 2.2. Applying straightforward the formulas of Appendix 2 we infer

$$N_\Gamma \wedge \mathbb{E} = E_\alpha \mathrm{d}x^3 \mathrm{d}x^\alpha, \quad \mathsf{int}(N_\Gamma)\mathbb{E} = E_3, \quad \mathsf{int}(N_\Gamma)\mathrm{d}\mathbb{E} = (\partial_3 E_\alpha - \partial_\alpha E_3)\, \mathrm{d}x^\alpha.$$

Hence transmission conditions (9) write for $h \in \{0, \varepsilon\}$

$$[E_\alpha]_{x_3=h} = 0, \quad \left[\frac{1}{\mu}\left(\partial_3 E_\alpha - \partial_\alpha E_3\right)\right]_{x_3=h} = 0, \quad [q\, E_3]_{x_3=h} = 0. \tag{22}$$

3.3 Rescaling in the thin layer

Denote by E_j^{e} and by E_j^{c} the respective restrictions of E_j to $\mathscr{V}_{\mathsf{e}}^{\varepsilon}$ and to $\mathscr{V}_{\mathsf{c}}$. In $\mathcal{O}_{\mathsf{m}}^{\varepsilon}$ we perform the rescaling $x_3 = \varepsilon\eta$, $\eta \in (0,1)$, and we denote by E_j^{m}, by g_{ij}^{m} and by g^{m} the following functions

$$\forall \eta \in (0,1), \quad \begin{cases} \mathcal{E}_j^{\mathsf{m}}(\mathbf{x}_{\mathsf{T}}, \eta) = E_j(\mathbf{x}_{\mathsf{T}}, \varepsilon\eta) \\ g_{ij}^{\mathsf{m}}(\mathbf{x}_{\mathsf{T}}, \eta) = g_{ij}(\mathbf{x}_{\mathsf{T}}, \varepsilon\eta), \quad \text{for } i,j = 1,2,3 \\ g^{\mathsf{m}}(\mathbf{x}_{\mathsf{T}}, \eta) = g(\mathbf{x}_{\mathsf{T}}, \varepsilon\eta) \end{cases}.$$

Observe that $g_{\alpha\beta}^{\mathsf{m}}(\mathbf{x}_{\mathsf{T}}, \eta) = g_{\alpha\beta}^{0}(\mathbf{x}_{\mathsf{T}}) - 2\varepsilon\eta b_{\alpha\beta}(\mathbf{x}_{\mathsf{T}}) + \varepsilon^2\eta^2 c_{\alpha\beta}(\mathbf{x}_{\mathsf{T}})$, hence for $l \in \mathbb{N}$, $\partial_\eta^l g_{\alpha\beta}^{\mathsf{m}} = O(\varepsilon^l)$, while $\partial_\alpha^l g_{\iota\kappa}^{\mathsf{m}} = O(1)$. Denote by

$$\delta \mathrm{d}\mathbb{E} = a_i^{\mathsf{m}}(\mathbf{x}_{\mathsf{T}}, \eta)\mathrm{d}x^i, \text{in } \mathcal{O}_{\mathsf{m}}^{\varepsilon}.$$

Applying formula (2.09) with the metric given by (20), and performing the rescaling $x_3 = \varepsilon\eta$, we infer,

$$\begin{aligned} a_\lambda^{\mathsf{m}} = &-\frac{1}{\varepsilon^2}\partial_\eta^2\mathcal{E}_\lambda^{\mathsf{m}} + \frac{1}{\varepsilon}\left(\partial_\eta\partial_\lambda\mathcal{E}_3^{\mathsf{m}} + \epsilon_{\alpha\beta3}\epsilon_{\iota\kappa3}\frac{g_{\lambda\iota}^{\mathsf{m}}}{\sqrt{g^{\mathsf{m}}}}\frac{\partial_\eta}{\varepsilon}\left(\frac{g_{\alpha\kappa}^{\mathsf{m}}}{\sqrt{g^{\mathsf{m}}}}\right)\partial_\eta\mathcal{E}_\beta^{\mathsf{m}}\right) \\ &+ \epsilon_{\alpha\beta3}\epsilon_{\iota\kappa3}\frac{g_{\lambda\iota}^{\mathsf{m}}}{\sqrt{g^{\mathsf{m}}}}\left(\partial_\kappa\left(\frac{1}{\sqrt{g^{\mathsf{m}}}}\partial_\alpha\mathcal{E}_\beta^{\mathsf{m}}\right) - \frac{\partial_\eta}{\varepsilon}\left(\frac{g_{\alpha\kappa}^{\mathsf{m}}}{\sqrt{g^{\mathsf{m}}}}\right)\partial_\beta\mathcal{E}_3^{\mathsf{m}}\right), \end{aligned} \tag{23}$$

$$a_3^{\mathsf{m}} = \frac{1}{\varepsilon}\epsilon_{\alpha\beta3}\epsilon_{\iota\kappa3}\partial_\kappa\left(\frac{g_{\alpha\iota}^{\mathsf{m}}}{\sqrt{g^{\mathsf{m}}}}\partial_\eta\mathcal{E}_\beta^{\mathsf{m}}\right) + \epsilon_{\alpha\beta3}\epsilon_{\iota\kappa3}\partial_\alpha\left(\frac{g_{\beta\iota}^{\mathsf{m}}}{\sqrt{g^{\mathsf{m}}}}\partial_\kappa\mathcal{E}_3^{\mathsf{m}}\right). \tag{24}$$

The divergence free condition $\delta\mathbb{E}^{\mathsf{m}} = 0$ with equality (2.07) writes then

$$\frac{1}{\varepsilon}\partial_\eta\mathcal{E}_3^{\mathsf{m}} + \frac{1}{\sqrt{g^{\mathsf{m}}}}\frac{\partial_\eta}{\varepsilon}(\sqrt{g^{\mathsf{m}}})\mathcal{E}_3^{\mathsf{m}} + \epsilon_{\alpha\beta3}\epsilon_{\iota\kappa3}\frac{1}{\sqrt{g^{\mathsf{m}}}}\partial_\alpha\left(\frac{g_{\kappa\beta}^{\mathsf{m}}}{\sqrt{g^{\mathsf{m}}}}\mathcal{E}_\iota^{\mathsf{m}}\right) = 0. \tag{25}$$

The transmission conditions (22) in $x_3 = \varepsilon$ become

$$\frac{1}{\mu_{\mathsf{e}}}\left(\partial_3 E_\lambda - \partial_\lambda E_3\right)|_{x_3=\varepsilon^+} = \frac{1}{\mu_{\mathsf{m}}}\left(\frac{1}{\varepsilon}\partial_\eta\mathcal{E}_\lambda^{\mathsf{m}} - \partial_\lambda\mathcal{E}_3^{\mathsf{m}}\right)\Big|_{\eta=1} \tag{26a}$$

$$E_\lambda|_{x_3=\varepsilon^+} = \mathcal{E}_\lambda^{\mathsf{m}}|_{\eta=1}. \tag{26b}$$

The transmission conditions (22) in $x_3 = 0$ write

$$\frac{1}{\mu_{\mathsf{m}}}\left(\frac{1}{\varepsilon}\partial_\eta\mathcal{E}_\lambda^{\mathsf{m}} - \partial_\lambda\mathcal{E}_3^{\mathsf{m}}\right)\Big|_{\eta=0} = \frac{1}{\mu_{\mathsf{c}}}\left(\partial_3 E_\lambda - \partial_\lambda E_3\right)|_{x_3=0^-} \tag{27a}$$

$$\mathcal{E}_\lambda^{\mathsf{m}}|_{\eta=0} = E_\lambda|_{x_3=0^-}, \tag{27b}$$

and the transmission conditions for the normal components E_3 are

$$q_{\mathsf{e}}E_3|_{x_3=\varepsilon^+} = q_{\mathsf{m}}\mathcal{E}_3^{\mathsf{m}}|_{\eta=1}, \quad q_{\mathsf{m}}\mathcal{E}_3^{\mathsf{m}}|_{\eta=0} = q_{\mathsf{c}}E_3|_{x_3=0^-}. \tag{28}$$

4 Ansatz and formal expansion

We set now our ansatz. We look solutions written as formal series in ε

$$\mathbb{E}|_{\mathcal{O}_{\mathsf{e}}^{\varepsilon}} = \widetilde{\mathbb{E}}^{\mathsf{e},0}|_{\mathcal{O}_{\mathsf{e}}^{\varepsilon}} + \varepsilon\widetilde{\mathbb{E}}^{\mathsf{e},1}|_{\mathcal{O}_{\mathsf{e}}^{\varepsilon}} + \cdots, \text{ in } \mathcal{O}_{\mathsf{e}}^{\varepsilon}, \tag{29a}$$

$$\mathbb{E}|_{\mathcal{O}_{\mathsf{c}}} = \mathbb{E}^{\mathsf{c},0} + \varepsilon\mathbb{E}^{\mathsf{c},1} + \cdots, \text{ in } \mathcal{O}_{\mathsf{c}}, \tag{29b}$$

and in the cylinder $\Gamma \times (0,1)$,

$$\mathbb{E}|_{\mathcal{O}^{\varepsilon}_{\mathsf{m}}} \circ \Phi(\mathbf{x}_{\mathsf{T}}, \varepsilon\eta) = \mathcal{E}^{\mathsf{m},0}(\mathbf{x}_{\mathsf{T}}, \eta) + \varepsilon \mathcal{E}^{\mathsf{m},1}(\mathbf{x}_{\mathsf{T}}, \eta) + \cdots, \tag{29c}$$

where the 1–forms $(\widetilde{\mathbb{E}}^{\mathsf{e},n})_{n\in\mathbb{N}}$, and $(\mathbb{E}^{\mathsf{c},n})_{n\in\mathbb{N}}$ are defined in ε–independent domains. We emphasize that the sequence $(\widetilde{\mathbb{E}}^{\mathsf{e},n})_{n\in\mathbb{N}}$ is defined in $(\mathcal{O}^{\varepsilon}_{\mathsf{m}})^{\mathbb{N}}$ even if its associated serie does not approach $\mathbb{E}$ in the thin layer.

Remark 4.1. The 1–forms $(\mathcal{E}^{\mathsf{m},n})_{n\in\mathbb{N}}$ are profiles defined in the cylinder $\Gamma \times (0,1)$; note the difference with the 1–forms $(\mathbb{E}^{\mathsf{c},n})_{n\in\mathbb{N}}$ and $(\widetilde{\mathbb{E}}^{\mathsf{e},n})_{n\in\mathbb{N}}$. These profiles are the key-point of the following asymptotic expansion. △

In $\mathscr{V}_{\Gamma}$, for $n \in \mathbb{N}$, we denote by

$$\widetilde{\mathbb{E}}^{\mathsf{e},n} = \widetilde{E}^{\mathsf{e},n}_i(\mathbf{x}_{\mathsf{T}}, x_3)\mathrm{d}x^i, \quad \mathbb{E}^{\mathsf{c},n} = E^{\mathsf{c},n}_i(\mathbf{x}_{\mathsf{T}}, x_3)\mathrm{d}x^i,$$
$$\mathcal{E}^{\mathsf{m},n} = \mathcal{E}^{\mathsf{m},n}_i(\mathbf{x}_{\mathsf{T}}, \eta)\mathrm{d}x^i, \quad \eta = x_3/\varepsilon.$$

Our aim is to identify the first two terms of the sequences and to estimate the remainder term. Suppose that for $n \in \mathbb{N}$, the forms $\left(\widetilde{E}^{\mathsf{e},n}_k\right)_{k=1,2,3}$ are as regular as necessary. Using formal Taylor expansion, we infer for $l = 0,1$

$$\partial^l_j \widetilde{E}^{\mathsf{e},n}_k|_{x_3=\varepsilon^+} = \partial^l_j \widetilde{E}^{\mathsf{e},n}_k|_{x_3=0^+} + \varepsilon\partial_3\partial^l_j \widetilde{E}^{\mathsf{e},n}_k|_{x_3=0^+} + \cdots \tag{30}$$

It is convenient to define $\mathbb{E}^n$ for $n \in \mathbb{N}$ by

$$\mathbb{E}^n = \widetilde{\mathbb{E}}^{\mathsf{e},n}, \text{ in } \mathcal{O}_{\mathsf{e}}, \quad \mathbb{E}^n = \mathbb{E}^{\mathsf{c},n}, \text{ in } \mathcal{O}_{\mathsf{c}}.$$

We are now ready to derive formally our asymptotics. Replace the coefficients $(\mathcal{E}^{\mathsf{m}}_j)_{j=1,\cdots 3}$ and $(E_j)_{j=1,\cdots 3}$ in equations (23)–(24)–(25) and in transmission conditions (26)–(27)–(28) by their respective formal expansion (29), and use the formal Taylor expansion (30). Observe that for any $n \in \mathbb{N}$, we necessarily have

$$\delta\mathrm{d}\mathbb{E}^n - \tilde{\mu}\tilde{q}\mathbb{E}^n = \delta^n_0\,\mathbb{J}, \text{ in } \mathcal{O}_{\mathsf{e}} \cup \mathcal{O}_{\mathsf{c}}, \quad N_{\partial\mathcal{O}} \wedge \mathbb{E}^n|_{\partial\mathcal{O}} = 0, \text{ on } \partial\mathcal{O}. \tag{31a}$$

$$\text{Observe that} \quad \delta\mathbb{E}^n = 0, \quad \text{in } \mathcal{O}_{\mathsf{c}} \cup \mathcal{O}_{\mathsf{e}}, \tag{31b}$$

since $\delta\mathbb{J} = 0$. It remains to build the appropriate transmission conditions by identifying the terms with the same power of ε.

4.1 Order 0

The term of order -2 in (23) vanishes hence $\partial^2_\eta \mathcal{E}^{\mathsf{m},0}_\alpha = 0$. From the divergence free condition (25) we infer $\partial_\eta \mathcal{E}^{\mathsf{m},0}_3 = 0$. Equality (26a) implies $\partial_\eta \mathcal{E}^{\mathsf{m},0}_\alpha = 0$. Therefore the coefficients $\mathcal{E}^{\mathsf{m},0}_j$ depend only on $\mathbf{x}_{\mathsf{T}}$. From (26b)–(27b)–(28) we infer for $n = 0,1$

$$\partial^n_\beta \widetilde{E}^{\mathsf{e},0}_\alpha|_{x_3=0^+} = \partial^n_\beta E^{\mathsf{c},0}_\alpha|_{x_3=0^-}, \tag{32a}$$

$$q_{\mathsf{e}}\partial^n_\beta \widetilde{E}^{\mathsf{e},0}_3|_{x_3=0^+} = q_{\mathsf{c}}\partial^n_\beta E^{\mathsf{c},0}_3|_{x_3=0^-}. \tag{32b}$$

4.2 Order 1

Since $\partial_\eta \mathcal{E}_\alpha^{\mathsf{m},0}$ and the terms of order -1 in (23) vanish, we infer

$$\partial_\eta^2 \mathcal{E}_\alpha^{\mathsf{m},1} = 0. \tag{33}$$

Hence $\partial_\eta \mathcal{E}_\alpha^{\mathsf{m},1}$ is constant with respect to η. Therefore, according to (26a)

$$\frac{1}{\mu_\mathsf{e}} \left(\partial_3 \widetilde{E}_\alpha^{\mathsf{e},0} - \partial_\alpha \widetilde{E}_3^{\mathsf{e},0} \right) |_{x_3=0^+} = \frac{1}{\mu_\mathsf{c}} \left(\partial_3 E_\alpha^{\mathsf{c},0} - \partial_\alpha E_3^{\mathsf{c},0} \right) |_{x_3=0^-}. \tag{34}$$

According to (31)–(32)–(34) the 1–forms $\widetilde{\mathbb{E}}^{\mathsf{e},0}$ and $\mathbb{E}^{\mathsf{c},0}$ satisfy the elliptic problem (10). According to (27b) and to (28), we infer

$$\mathcal{E}_\alpha^{\mathsf{m},0}(\mathbf{x}_\mathsf{T}, \eta) = E_\alpha^{\mathsf{c},0}(\mathbf{x}_\mathsf{T}, 0), \tag{35a}$$

$$\mathcal{E}_3^{\mathsf{m},0}(\mathbf{x}_\mathsf{T}, \eta) = \frac{q_\mathsf{c}}{q_\mathsf{m}} E_3^{\mathsf{c},0}(\mathbf{x}_\mathsf{T}, 0). \tag{35b}$$

Therefore the terms of order 0 are entirely determined. According to (27a), using (35) and since $\partial_\eta \mathcal{E}_\alpha^{\mathsf{m},1}$ does not depend on η accordingly (33), we infer

$$\partial_\eta \mathcal{E}_\alpha^{\mathsf{m},1}(\mathbf{x}_\mathsf{T}, \eta) = \frac{q_\mathsf{c}}{q_\mathsf{m}} \partial_\alpha E_3^{\mathsf{c},0}|_{x_3=0^-} + \frac{\mu_\mathsf{m}}{\mu_\mathsf{c}} \left(\partial_3 E_\alpha^{\mathsf{c},0} - \partial_\alpha E_3^{\mathsf{c},0} \right) |_{x_3=0^-}. \tag{36}$$

The transmission conditions follow

$$\widetilde{E}_\alpha^{\mathsf{e},1}|_{x_3=0^+} + \partial_3 \widetilde{E}_\alpha^{\mathsf{e},0}|_{x_3=0^+} = \partial_\eta \mathcal{E}_\alpha^{\mathsf{m},1} + \mathcal{E}_\alpha^{\mathsf{m},1}|_{\eta=0},$$

and

$$\mathcal{E}_\alpha^{\mathsf{m},1}|_{\eta=0} = E_\alpha^{\mathsf{c},1}|_{x_3=0^-}.$$

Therefore we infer

$$\widetilde{E}_\alpha^{\mathsf{e},1}|_{x_3=0^+} - E_\alpha^{\mathsf{c},1}|_{x_3=0^-} = \partial_\eta \mathcal{E}_\alpha^{\mathsf{m},1} - \partial_3 \widetilde{E}_\alpha^{\mathsf{e},0}|_{x_3=0^+}.$$

Using (36) and according to (32) and (34) we infer

$$\begin{aligned} \widetilde{E}_\alpha^{\mathsf{e},1}|_{x_3=0^+} - E_\alpha^{\mathsf{c},1}|_{x_3=0^-} = &\left(\frac{q_\mathsf{c}}{q_\mathsf{m}} - \frac{q_\mathsf{c}}{q_\mathsf{e}} \right) \partial_\alpha E_3^{\mathsf{c},0}|_{x_3=0^-} \\ &+ \frac{\mu_\mathsf{m} - \mu_\mathsf{e}}{\mu_\mathsf{c}} \left(\partial_3 E_\alpha^{\mathsf{c},0} - \partial_\alpha E_3^{\mathsf{c},0} \right) |_{x_3=0^-}. \end{aligned} \tag{37}$$

The divergence free condition leads to

$$\partial_\eta \mathcal{E}_3^{\mathsf{m},1} = -\frac{1}{\sqrt{g^0}} \epsilon_{\alpha\beta3} \epsilon_{\iota\kappa3} \partial_\alpha \left(\frac{g^0_{\kappa\beta}}{\sqrt{g^0}} E_\iota^{\mathsf{c},0} \right) |_{x_3=0^-} + 2\mathscr{H} \frac{q_\mathsf{c}}{q_\mathsf{m}} E_3^{\mathsf{c},0}|_{x_3=0^-}, \tag{38}$$

where $\mathscr{H}$ is given by (21). Transmission condition (28) implies

$$q_\mathsf{e} \widetilde{E}_3^{\mathsf{e},1}|_{x_3=0^+} + q_\mathsf{e} \partial_3 \widetilde{E}_3^{\mathsf{e},0}|_{x_3=0^+} = q_\mathsf{m} \partial_\eta \mathcal{E}_3^{\mathsf{m},1} + q_\mathsf{c} E_3^{\mathsf{c},1}|_{x_3=0^-}. \tag{39}$$

According to (10) $\mathbb{E}^{\mathsf{c},0}$ satisfy the divergence free condition hence

$$-\frac{1}{\sqrt{g^0}} \epsilon_{\alpha\beta3} \epsilon_{\iota\kappa3}\, \partial_\alpha \left(\frac{g^0_{\kappa\beta}}{\sqrt{g^0}} E_\iota^{\mathsf{c},0} \right) \Bigg|_{x_3=0^-} = \partial_3 E_3^{\mathsf{c},0}|_{x_3=0^-} - 2\mathscr{H} E_3^{\mathsf{c},0}|_{x_3=0^-}, \tag{40}$$

and similarly for $\widetilde{\mathbb{E}}^{\mathsf{e},0}$ by replacing $E_i^{\mathsf{c},0}$ by $\widetilde{E}_i^{\mathsf{e},0}$. From (38)–(40) we infer

$$\partial_\eta \mathcal{E}_3^{\mathsf{m},1} = \partial_3 E_3^{\mathsf{c},0}|_{x_3=0^-} + 2\mathscr{H}\left(\frac{q_\mathsf{c}}{q_\mathsf{m}} - 1\right) E_3^{\mathsf{c},0}|_{x_3=0^-}. \tag{41}$$

Moreover using (32) in (40) we infer

$$q_\mathsf{e}\partial_3 \widetilde{E}_3^{\mathsf{e},0}|_{x_3=0^+} = q_\mathsf{e}\partial_3 E_3^{\mathsf{c},0}|_{x_3=0^-} - 2\mathscr{H}(q_\mathsf{e} - q_\mathsf{c}) E_3^{\mathsf{c},0}|_{x_3=0^-},$$

and therefore (39) with equality (21) implies

$$q_\mathsf{e}\widetilde{E}_3^{\mathsf{e},1}|_{x_3=0^+} - q_\mathsf{c} E_3^{\mathsf{c},1}|_{x_3=0^-} = (q_\mathsf{m} - q_\mathsf{e})\,\frac{1}{\sqrt{g|_{x_3=0}}}\,\partial_3\left(\sqrt{g}\widetilde{E}_3^{\mathsf{e},0}\right)\Big|_{x_3=0^+}. \tag{42}$$

4.3 Order 2

Since $\partial_\eta \mathcal{E}_\alpha^{\mathsf{m},0} = 0$ we identify the terms in ε^2 in (23) to infer

$$\begin{aligned}\partial_\eta^2 \mathcal{E}_\lambda^{\mathsf{m},2} &= \partial_\eta\partial_\lambda \mathcal{E}_3^{\mathsf{m},1} + \epsilon_{\alpha\beta3}\epsilon_{\iota\kappa3}\frac{g^0_{\lambda\iota}}{\sqrt{g^0}}\left\{\frac{\partial_\eta}{\varepsilon}\left(\frac{g^\mathsf{m}_{\alpha\kappa}}{\sqrt{g^\mathsf{m}}}\right)\Big|_{\eta=0}\partial_\eta \mathcal{E}_\beta^{\mathsf{m},1}\right.\\ &\left.+\left(\partial_\kappa\left(\frac{1}{\sqrt{g^0}}\partial_\alpha \mathcal{E}_\beta^{\mathsf{m},0}\right) - \frac{\partial_\eta}{\varepsilon}\left(\frac{g^\mathsf{m}_{\alpha\kappa}}{\sqrt{g^\mathsf{m}}}\right)\Big|_{\eta=0}\partial_\beta \mathcal{E}_3^{\mathsf{m},0}\right)\right\} + \mu_\mathsf{m} q_\mathsf{m}\sqrt{g^0}\mathcal{E}_\lambda^{\mathsf{m},0}.\end{aligned} \tag{43}$$

Since the right-hand side of the previous equality does not depend on η, we have

$$\begin{aligned}&\frac{1}{\mu_\mathsf{e}}\left(\partial_3\widetilde{E}_\lambda^{\mathsf{e},1} - \partial_\lambda\widetilde{E}_3^{\mathsf{e},1}\right)\Big|_{x_3=0^+} - \frac{1}{\mu_\mathsf{c}}\left(\partial_3 E_\lambda^{\mathsf{c},1} - \partial_\lambda E_3^{\mathsf{c},1}\right)\Big|_{x_3=0^-} = \frac{1}{\mu_\mathsf{m}}\Bigg(\partial_\eta^2\mathcal{E}_\lambda^{\mathsf{m},2}\\ &- \partial_\lambda\partial_\eta\mathcal{E}_3^{\mathsf{m},1}\Bigg) - \frac{1}{\mu_\mathsf{e}}\left(\partial_3^2\widetilde{E}_\lambda^{\mathsf{e},0}|_{x_3=0^+} - \partial_\lambda\partial_3\widetilde{E}_3^{\mathsf{e},0}|_{x_3=0^+}\right).\end{aligned}$$

Since $\delta\mathrm{d}\widetilde{\mathbb{E}}^{\mathsf{e},0} - \mu_\mathsf{e} q_\mathsf{e}\widetilde{\mathbb{E}}^{\mathsf{e},0} = 0$, explicit formulae of Appendix 2 imply

$$\begin{aligned}\partial_3^2\widetilde{E}_j^{\mathsf{e},0}|_{x_3=0^+} = \Bigg[&\mu_\mathsf{e} q_\mathsf{e}\sqrt{g}\widetilde{E}_j^{\mathsf{e},0} + \partial_3\partial_\lambda\widetilde{E}_3^{\mathsf{e},0} + \epsilon_{\alpha\beta3}\epsilon_{\iota\kappa3}\frac{g_{\lambda\iota}}{\sqrt{g}}\partial_3\left(\frac{g_{\alpha\kappa}}{\sqrt{g}}\right)\partial_3\widetilde{E}_\beta^{\mathsf{e},0}\\ &+ \epsilon_{\alpha\beta3}\epsilon_{\iota\kappa3}\frac{g_{\lambda\iota}}{\sqrt{g}}\left(\partial_\kappa\left(\frac{1}{\sqrt{g}}\partial_\alpha\widetilde{E}_\beta^{\mathsf{e},0}\right) - \partial_3\left(\frac{g_{\alpha\kappa}}{\sqrt{g}}\right)\partial_\alpha\widetilde{E}_3^{\mathsf{e},0}\right)\Bigg]\Bigg|_{x_3=0^+}.\end{aligned}$$

According to the transmission condition at the order 0, the following equalities hold

$$\begin{aligned}&\frac{1}{\mu_\mathsf{e}}\left(\partial_\lambda\widetilde{E}_3^{\mathsf{e},0} - \partial_3\widetilde{E}_\lambda^{\mathsf{e},0}\right)|_{x_3=0^+} = \frac{1}{\mu_\mathsf{m}}\left(\partial_\lambda\mathcal{E}_3^{\mathsf{m},0} - \partial_\eta\mathcal{E}_\lambda^{\mathsf{m},1}\right)|_{\eta=0},\\ &\widetilde{E}_\lambda^{\mathsf{e},0}|_{x_3=0^+} = \mathcal{E}_\lambda^{\mathsf{m},0}|_{\eta=0},\end{aligned}$$

hence we infer the following transmission conditions

$$\begin{aligned}\frac{1}{\mu_\mathsf{e}}&\left(\partial_3\widetilde{E}_\lambda^{\mathsf{e},1} - \partial_\lambda\widetilde{E}_3^{\mathsf{e},1}\right) - \frac{1}{\mu_\mathsf{c}}\left(\partial_3 E_\lambda^{\mathsf{c},1} - \partial_\lambda E_3^{\mathsf{c},1}\right) = (q_\mathsf{m} - q_\mathsf{e})\,\widetilde{E}_\lambda^{\mathsf{e},0}|_{x_3=0^+}\\ &+ \left(\frac{1}{\mu_\mathsf{m}} - \frac{1}{\mu_\mathsf{e}}\right)\epsilon_{\alpha\beta3}\epsilon_{\iota\kappa3}\frac{g_{\lambda\alpha}}{\sqrt{g}}\partial_\beta\left(\frac{1}{\sqrt{g}}\partial_\iota\widetilde{E}_\kappa^{\mathsf{e},0}\right)\Big|_{x_3=0^+}.\end{aligned} \tag{44}$$

Therefore $\mathbb{E}^1$ satisfies (31) for $n = 1$ with the transmission conditions (37)–(44) written in local coordinates. Using equalities (36)–(41) we infer

$$\mathcal{E}_\lambda^{\mathsf{m},1}(\mathsf{x}_\mathsf{T}, \eta) = \eta \partial_\eta \mathcal{E}_\lambda^{\mathsf{m},1} + E_\lambda^{\mathsf{c},1}|_{x_3=0^-}, \quad \mathcal{E}_3^{\mathsf{m},1}(\mathsf{x}_\mathsf{T}, \eta) \quad = \eta \partial_\eta \mathcal{E}_3^{\mathsf{m},1} + \frac{q_\mathsf{c}}{q_\mathsf{m}} E_3^{\mathsf{c},1}|_{x_3=0^-}.$$

Remark 4.2. The coefficients at the order 1 are now uniquely determined. Since

$$\partial_\eta \mathcal{E}_\alpha^{\mathsf{m},2}|_{\eta=0} = \partial_\alpha \mathcal{E}_3^{\mathsf{m},1}|_{\eta=0} - \frac{\mu_\mathsf{m}}{\mu_\mathsf{c}} \left(\partial_\alpha E_3^{\mathsf{c},1} - \partial_3 E_\alpha^{\mathsf{c},1} \right) |_{x_3=0^-},$$

$\partial_\eta \mathcal{E}_\lambda^{\mathsf{m},2}$ is uniquely determined by (43)

$$\partial_\eta \mathcal{E}_\lambda^{\mathsf{m},2} = \eta \partial_\eta^2 \mathcal{E}_\lambda^{\mathsf{m},2} + \partial_\alpha \mathcal{E}_3^{\mathsf{m},1}|_{\eta=0} - \frac{\mu_\mathsf{m}}{\mu_\mathsf{c}} \left(\partial_\alpha E_3^{\mathsf{c},1} - \partial_3 E_\alpha^{\mathsf{c},1} \right) |_{x_3=0^-}. \tag{45}$$

△

Remark 4.3. Transmission condition (42) might be obtained straightforward from (31)–(37)–(44). Writing $\delta \mathrm{d} \widetilde{\mathbb{E}}^{\mathsf{e},1} = \widetilde{a}_i^{\mathsf{e},1} \mathrm{d}x^i$ and $\delta \mathrm{d} \mathbb{E}^{\mathsf{c},1} = a_i^{\mathsf{c},1} \mathrm{d}x^i$ we infer

$$a_3^{\mathsf{c},1} = \frac{1}{\sqrt{g}} \epsilon_{\alpha\beta 3} \epsilon_{\iota\kappa 3} \partial_\alpha \left(\frac{g_{\beta\kappa}}{\sqrt{g}} \left(\partial_3 E_\iota^{\mathsf{c},1} - \partial_\iota E_3^{\mathsf{c},1} \right) \right),$$

and similarly for $\widetilde{a}_3^{\mathsf{e},1}$ by replacing $E^{\mathsf{c},1}$ by $\widetilde{E}^{\mathsf{e},1}$. According to (44) we have

$$\frac{1}{\mu_\mathsf{e}} \widetilde{a}_3^{\mathsf{e},1}|_{x_3=0^+} - \frac{1}{\mu_\mathsf{c}} a_3^{\mathsf{c},1}|_{x_3=0^-} = \frac{(q_\mathsf{m} - q_\mathsf{e})}{\sqrt{g}} \epsilon_{\alpha\beta 3} \epsilon_{\iota\kappa 3} \partial_\alpha \left(\frac{g_{\beta\kappa}}{\sqrt{g}} \widetilde{E}_\iota^{\mathsf{e},1} \right) \bigg|_{x_3=0^+}.$$

The divergence free property of $\widetilde{\mathbb{E}}^{\mathsf{e},0}$ applied in $x_3 = 0^+$ implies

$$\frac{1}{\mu_\mathsf{e}} a_3^{\mathsf{e},1}|_{x_3=0^+} - \frac{1}{\mu_\mathsf{c}} a_3^{\mathsf{c},1}|_{x_3=0^-} = -(q_\mathsf{m} - q_\mathsf{e}) \frac{1}{\sqrt{g|_{x_3=0}}} \partial_3 \left(\sqrt{g} \widetilde{E}_3^{\mathsf{e},0} \right) |_{x_3=0^+}.$$

Moreover we have

$$\frac{1}{\mu_\mathsf{e}} a_3^{\mathsf{e},1}|_{x_3=0^+} + q_\mathsf{e} \widetilde{E}_3^{\mathsf{e},1}|_{x_3=0^+} = \frac{1}{\mu_\mathsf{c}} a_3^{\mathsf{c},1}|_{x_3=0^-} + q_\mathsf{c} E_3^{\mathsf{c},1}|_{x_3=0^-} = 0,$$

therefore, we infer

$$q_\mathsf{e} \widetilde{E}_3^{\mathsf{e},1}|_{x_3=0^+} - q_\mathsf{c} E_3^{\mathsf{c},1}|_{x_3=0^-} = (q_\mathsf{m} - q_\mathsf{e}) \frac{1}{\sqrt{g|_{x_3=0}}} \partial_3 \left(\sqrt{g} \widetilde{E}_3^{\mathsf{e},0} \right) \bigg|_{x_3=0^+},$$

which is exactly condition (42). △

5 Justification of the expansion

Let us rewrite the equations satisfied by the first two terms of the asymptotic expansion of $\mathbb{E}$ in terms of differential forms. Denote by $\mathbb{S}$ and $\mathbb{T}$ the following forms

$$\mathbb{S} = \left((q_\mathsf{m} - q_\mathsf{e}) \widetilde{E}_\lambda^{\mathsf{e},0}|_{x_3=0^+} + \left(\frac{1}{\mu_\mathsf{m}} - \frac{1}{\mu_\mathsf{e}} \right) \epsilon_{\alpha\beta 3} \epsilon_{\iota\kappa 3} \frac{g_{\lambda\alpha}}{\sqrt{g}} \partial_\beta \left(\frac{1}{\sqrt{g}} \partial_\iota \widetilde{E}_\kappa^{\mathsf{e},0}|_{x_3=0^+} \right) \right) \mathrm{d}x^\lambda$$

$$\mathbb{T} = \left(\left(\frac{q_\mathsf{c}}{q_\mathsf{m}} - \frac{q_\mathsf{c}}{q_\mathsf{e}} \right) \partial_\alpha E_3^{\mathsf{c},0}|_{x_3=0^-} + \frac{\mu_\mathsf{m} - \mu_\mathsf{e}}{\mu_\mathsf{c}} \left(\partial_3 E_\alpha^{\mathsf{c},0} - \partial_\alpha E_3^{\mathsf{c},0} \right) |_{x_3=0^-} \right) \mathrm{d}x^\alpha.$$

The reader easily verifies that the definitions (13)–(14) coincide with the above expressions of $\mathbb{S}$ and $\mathbb{T}$. The 1–form $\mathbb{E}^0$ satisfies (10) in a weak sense and $\mathbb{E}^1$ satisfy (31) with the following transmission conditions on Γ accordingly (37)–(42)

$$\frac{1}{\mu_{\mathsf{e}}}\mathsf{int}(N_\Gamma)\mathrm{d}\widetilde{\mathbb{E}}^{\mathsf{e},1}|_{\Gamma^+} - \frac{1}{\mu_{\mathsf{c}}}\mathsf{int}(N_\Gamma)\mathrm{d}\mathbb{E}^{\mathsf{c},1}|_{\Gamma^-} = \mathbb{S}, \tag{46a}$$

$$N_\Gamma \wedge \widetilde{\mathbb{E}}^{\mathsf{e},1}|_{\Gamma^+} - N_\Gamma \wedge \mathbb{E}^{\mathsf{c},1}|_{\Gamma^-} = N_\Gamma \wedge \mathbb{T}. \tag{46b}$$

Observe[7] that accordingly (42)

$$\delta\mathbb{S} = -(q_{\mathsf{m}} - q_{\mathsf{e}})\,\frac{1}{\sqrt{g|_{x_3=0}}}\,\partial_3\left(\sqrt{g}\widetilde{E}_3^{\mathsf{e},0}\right)\Big|_{x_3=0^+}. \tag{47}$$

In the cylinder $\Gamma \times (0,1)$, the 1–form $\mathcal{E}^{\mathsf{m},0}$ equals

$$\mathcal{E}^{\mathsf{m},0} = E_\alpha^{\mathsf{c},0}|_{x_3=0^-}\,\mathrm{d}x^\alpha + \frac{q_{\mathsf{c}}}{q_{\mathsf{m}}}E_3^{\mathsf{c},0}|_{x_3=0^-}\,\mathrm{d}x^3, \tag{48}$$

while the 1–form $\mathcal{E}^{\mathsf{m},1}$ equals

$$\begin{aligned}\mathcal{E}^{\mathsf{m},1} = &\left\{E_\alpha^{\mathsf{c},1}|_{x_3=0^-} + \eta\left(\frac{q_{\mathsf{c}}}{q_{\mathsf{m}}}\partial_\alpha E_3^{\mathsf{c},0} + \frac{\mu_{\mathsf{m}}}{\mu_{\mathsf{c}}}\left(\partial_3 E_\alpha^{\mathsf{c},0} - \partial_\alpha E_3^{\mathsf{c},0}\right)\right)\Big|_{x_3=0^-}\right\}\mathrm{d}x^\alpha \\ &+ \left\{\frac{q_{\mathsf{c}}}{q_{\mathsf{m}}}E_3^{\mathsf{c},1}|_{x_3=0^-} + \eta\left(\partial_3 E_3^{\mathsf{c},0} + 2\mathscr{H}\left(\frac{q_{\mathsf{m}}}{q_{\mathsf{c}}} - 1\right)E_3^{\mathsf{c},0}\right)\Big|_{x_3=0^-}\right\}\mathrm{d}x^3.\end{aligned} \tag{49}$$

5.1 Regularity results

We present now the regularity of the 1-forms $\mathbb{E}^0$ and $\mathbb{E}^1$.

Proposition 5.1. *Let Hypothesis 2.2 hold. Moreover let $s \geq 0$ and $\mathbb{J}$ belong to $H^{1+s}\Omega^1(\mathcal{O}_{\mathsf{e}}^{d_0})$. Then the 1-forms $\mathbb{E}^0$ and $\mathbb{E}^1$ exist and are unique. Moreover the following regularity results hold*

$$\begin{aligned}\widetilde{\mathbb{E}}^{\mathsf{e},0} \in H^{3+s}\Omega^1(\mathcal{O}_{\mathsf{e}}), &\quad \mathbb{E}^{\mathsf{c},0} \in H^{3+s}\Omega^1(\mathcal{O}_{\mathsf{c}}), \\ \widetilde{\mathbb{E}}^{\mathsf{e},1} \in H^{2+s}\Omega^1(\mathcal{O}_{\mathsf{e}}), &\quad \mathbb{E}^{\mathsf{c},1} \in H^{2+s}\Omega^1(\mathcal{O}_{\mathsf{c}}).\end{aligned}$$

Proof. All the assertions concerning $\mathbb{E}^0$ are proved in the above Proposition 2.6. Since $\widetilde{\mathbb{E}}^{\mathsf{e},0}$ and $\mathbb{E}^{\mathsf{c},0}$ belong respectively to $H^{3+s}\Omega^1(\mathcal{O}_{\mathsf{e}})$ and $H^{3+s}\Omega^1(\mathcal{O}_{\mathsf{c}})$, the forms $\mathbb{S}$ and $\mathbb{T}$ belong to the following Sobolev spaces

$$\mathbb{S} \in H^{1/2+s}\Omega^1(\Gamma), \quad \mathbb{T} \in H^{3/2+s}\Omega^1(\Gamma).$$

[7]Since $q_{\mathsf{e}}\mathsf{int}(N_\Gamma)\widetilde{\mathbb{E}}^{\mathsf{e},1}|_{\Gamma^+} = \mathsf{int}(N_\Gamma)\left((1/\mu_{\mathsf{e}})\delta\mathrm{d}\widetilde{\mathbb{E}}^{\mathsf{e},1}|_{\Gamma^+}\right)$ using (2.05) since $\mathrm{d}N_\Gamma = 0$ we infer $\mathsf{int}(N_\Gamma)\left((1/\mu_{\mathsf{e}})\delta\mathrm{d}\widetilde{\mathbb{E}}^{\mathsf{e},1}|_{\Gamma^+}\right) = -\delta\left((1/\mu_{\mathsf{e}})\mathsf{int}(N_\Gamma)\mathrm{d}\widetilde{\mathbb{E}}^{\mathsf{e},1}|_{\Gamma^+}\right)$, and similarly for $\mathbb{E}^{\mathsf{c},1}$. Therefore accordingly (46a) we infer $q_{\mathsf{e}}\mathsf{int}(N_\Gamma)\widetilde{\mathbb{E}}^{\mathsf{e},1}|_{\Gamma^+} - q_{\mathsf{c}}\mathsf{int}(N_\Gamma)\mathbb{E}^{\mathsf{c},1}|_{\Gamma^-} = -\delta\mathbb{S}$, hence(47) according to (42).

Moreover accordingly (47), $\delta\mathbb{S} \in H^{3/2+s}(\Gamma)$. Let $C \in H^{2+s}\Omega^1(\mathcal{O}_\mathsf{c})$ such that

$$\delta C = 0, \text{ in } \mathcal{O}_\mathsf{c}, \quad \begin{cases} N_\Gamma \wedge C|_\Gamma = N_\Gamma \wedge \mathbb{T}, \\ \dfrac{1}{\mu_\mathsf{c}} \mathsf{int}(N_\Gamma)\mathrm{d}C|_\Gamma = \mathbb{S}, \end{cases}, \begin{cases} q_\mathsf{c} \mathsf{int}(N_\Gamma)C|_\Gamma = \delta\mathbb{S}, \\ \delta(q_\mathsf{c} C|_\Gamma) = 0. \end{cases}$$

Observe that $\delta\mathrm{d}C - \mu_\mathsf{c} q_\mathsf{c} C$ belongs to $H^s\Omega^1(\mathcal{O}_\mathsf{c})$. Denote by $\mathbb{U}$ the following 1-form

$$\mathbb{U} = \widetilde{\mathbb{E}}^{1,e}, \text{ in } \mathcal{O}_\mathsf{e}, \quad \mathbb{U} = \mathbb{E}^{1,c} - C, \text{ in } \mathcal{O}_\mathsf{c}.$$

Then $\mathbb{U}$ satisfies

$$\begin{aligned} &\delta\mathrm{d}\mathbb{U} - \mu_\mathsf{e} q_\mathsf{e} \mathbb{U} = 0, \text{ in } \mathcal{O}_\mathsf{e}, \\ &\delta\mathrm{d}\mathbb{U} - \mu_\mathsf{c} q_\mathsf{c} \mathbb{U} = -\delta\mathrm{d}C + \mu_\mathsf{c} q_\mathsf{c} C, \text{ in } \mathcal{O}_\mathsf{c}, \\ &N_{\partial\mathcal{O}} \wedge \mathbb{U}|_{\partial\mathcal{O}} = 0, \end{aligned}$$

with the following homogeneous transmission conditions on Γ

$$[N_\Gamma \wedge \mathbb{U}]_\Gamma = 0, \quad \left[\frac{1}{\tilde{\mu}} \mathsf{int}(N_\Gamma)\mathrm{d}\mathbb{U}\right]_\Gamma = 0, \quad [\tilde{q}\, \mathsf{int}(N_\Gamma)\mathbb{U}]_\Gamma = 0.$$

Performing as we did in Proposition 2.6, we infer Proposition 5.1. □

The next Proposition give the regularity of the 1-form $\mathcal{E}^{\mathsf{m},0}$, $\mathcal{E}^{\mathsf{m},1}$ and $\mathcal{E}^{\mathsf{m},2}$. Its proof easily comes from Proposition 5.1 and from the explicit expressions of the component of $\mathcal{E}^{\mathsf{m},n}$, for $n = 0, 1, 2$, given in Section 4.

Proposition 5.2. *Let Hypothesis 2.2 hold. Moreover let $s \geq 0$ and suppose that $\mathbb{J}$ belongs to $H^{1+s}\Omega^1\left(\mathcal{O}_\mathsf{e}^{d_0}\right)$. By abuse of notations*[8]*, we define $\mathcal{E}^{\mathsf{m},2}$ using* (45) *by*

$$\mathcal{E}^{\mathsf{m},2} = \int_0^{x_3/\varepsilon} \partial_\eta \mathcal{E}_\alpha^{\mathsf{m},2} \mathrm{d}\eta \, \mathrm{d}x^\alpha.$$

Denote by $C^\infty\Omega^1\left([0,1], H^{5/2+s-n}\Omega^1(\Gamma)\right)$ is the space of the 1-forms, which are smooth in the normal variable η, and which belong to $H^{5/2+s-n}\Omega^1(\Gamma)$ at given $\eta \in [0,1]$.

Then for $n = 0, 1, 2$, $\mathcal{E}^{\mathsf{m},n} \in C^\infty\Omega^1\left([0,1], H^{5/2+s-n}\Omega^1(\Gamma)\right)$.

5.2 Convergence

Suppose that Hypothesis 2.2 holds, and let the source current density $\mathbb{J}$ belong to $H^3\Omega^1\left(\mathcal{O}_\mathsf{e}^{d_0}\right)$, with $\delta\mathbb{J} = 0$. It is convenient to define

$$\begin{aligned} &\mathbb{E}_{app}^\mathsf{e} = \widetilde{\mathbb{E}}^{\mathsf{e},0} + \varepsilon \widetilde{\mathbb{E}}^{\mathsf{e},1}, \text{ in } \mathcal{O}_\mathsf{e}^\varepsilon, \quad \mathbb{E}_{app}^\mathsf{c} = \mathbb{E}^{\mathsf{c},0} + \varepsilon \mathbb{E}^{\mathsf{c},1}, \text{ in } \mathcal{O}_\mathsf{c}, \\ &\forall (\mathsf{x}_\mathsf{T}, x_3) \in \Gamma \times (0, \varepsilon), \, \mathbb{E}_{app}^\mathsf{m} \circ \Phi(\mathsf{x}_\mathsf{T}, x_3) = \sum_{n=0}^{2} \varepsilon^n \mathcal{E}^{\mathsf{m},n}(\mathsf{x}_\mathsf{T}, x_3/\varepsilon), \end{aligned}$$

and let $\mathbb{E}_{app}$ equal to $\mathbb{E}_{app}^\mathsf{e}$ in $\mathcal{O}_e^\varepsilon$, $\mathbb{E}_{app}^\mathsf{c}$ in $\mathcal{O}_c$ and to $\mathbb{E}_{app}^\mathsf{m}$ in $\mathcal{O}_m^\varepsilon$. According to the construction of the coefficients $(\mathcal{E}^{\mathsf{m},n})_{n=0,1,2}$ and using Proposition 5.2, there exists a 1-form $\mathbb{G} \in C^\infty\Omega^1\left([0,1], H^{1/2}\Omega^1(\Gamma)\right)$, such that

$$\delta\mathrm{d}\mathbb{E}_{app}^\mathsf{m} - \mu_\mathsf{m} q_\mathsf{m} \mathbb{E}_{app}^\mathsf{m} = \varepsilon \mathbb{G} \circ \Phi^{-1}, \text{ in } \mathcal{O}_\mathsf{m}^\varepsilon,$$

[8]Since $\mathcal{E}^{\mathsf{m},2}$ vanishes in $x_3 = 0$, it is not the third coefficient of the profile in $\Gamma \times (0, 1)$.

and for an ε–independent constant $C>0$,

$$\sup_{\eta\in[0,1]} \|\mathbb{G}(.,\eta)\|_{H^{1/2}\Omega^1(\Gamma)} \leqslant C, \quad \sup_{\eta\in[0,1]} \|\delta\mathbb{G}(.,\eta)\|_{H^{3/2}(\Gamma)} \leqslant C.$$

Define $\mathbb{W}$ by $\mathbb{W}=\mathbb{E}-\mathbb{E}_{app}$ and denote by $\mathbb{W}^{\mathsf{e}}$, $\mathbb{W}^{\mathsf{m}}$ and $\mathbb{W}^{\mathsf{c}}$ the respective restrictions of $\mathbb{W}$ to $\mathcal{O}_{\mathsf{e}}^{\varepsilon}$, $\mathcal{O}_{\mathsf{m}}^{\varepsilon}$ and $\mathcal{O}_{\mathsf{c}}$. In local coordinates, $\mathbb{W}^{\mathsf{e}}=W_i^{\mathsf{e}}\mathrm{d}x^i$, $\mathbb{W}^{\mathsf{m}}=W_i^{\mathsf{m}}\mathrm{d}x^i$ and $\mathbb{W}^{\mathsf{c}}=W_i^{\mathsf{c}}\mathrm{d}x^i$. Theorem 2.8 is a straightforward corollary of the following result.

Theorem 5.3. *There exists an ε–independent constant $C>0$ such that*

$$\|\mathbb{W}^{\mathsf{e}}\|_{H\Omega^1(\mathrm{d},\delta,\mathcal{O}_{\mathsf{e}}^{\varepsilon})} + \sqrt{\varepsilon}\|\mathbb{W}^{\mathsf{m}}\|_{H\Omega^1(\mathrm{d},\delta,\mathcal{O}_{\mathsf{m}}^{\varepsilon})} + \|\mathbb{W}^{\mathsf{c}}\|_{H\Omega^1(\mathrm{d},\delta,\mathcal{O}_{\mathsf{c}})} \leqslant C\varepsilon^2.$$

Proof. The 1–form $\mathbb{W}$ satisfies

$$\delta\mathrm{d}\mathbb{W}-\mu q\mathbb{W}=\varepsilon 1_{\mathcal{O}_{\mathsf{m}}^{\varepsilon}}\mathbb{G}, \text{ in } \mathcal{O}_{\mathsf{e}}^{\varepsilon}\cup\mathcal{O}_{\mathsf{m}}^{\varepsilon}\cup\mathcal{O}_{\mathsf{c}}, \quad N_{\partial\mathcal{O}}\wedge\mathbb{W}^{\mathsf{e}}|_{\partial\mathcal{O}}=0, \quad \text{on } \partial\mathcal{O},$$

with the following transmission conditions for $\mathscr{S}\in\{\Gamma_\varepsilon,\Gamma\}$

$$[N_{\mathscr{S}}\wedge\mathbb{W}]_{\mathscr{S}} = -[N_{\mathscr{S}}\wedge\mathbb{E}_{app}]_{\mathscr{S}}, \tag{50a}$$

$$\left[\frac{1}{\mu}\mathsf{int}(N_{\mathscr{S}})\mathrm{d}\mathbb{W}\right]_{\mathscr{S}} = -\left[\frac{1}{\mu}\mathsf{int}(N_{\mathscr{S}})\mathrm{d}\mathbb{E}_{app}\right]_{\mathscr{S}}. \tag{50b}$$

Let $\mathbb{E}_{app}^{\mathsf{e}}=E_i^{\mathsf{e},app}\mathrm{d}x^i$. Accordingly Proposition 5.1 $\mathbb{E}_{app}^{\mathsf{e}}\in H^4\Omega^1(\mathcal{O}_{\mathsf{e}})$. Hence there exists $f_\alpha\in H^{1/2}(\Gamma)$ and $g_j\in H^{3/2}(\Gamma)$ such that

$$\left(\partial_3E_\alpha^{\mathsf{e},app}-\partial_\alpha E_3^{\mathsf{e},app}\right)|_{x_3=\varepsilon} = \sum_{l=0,1}\varepsilon^l\partial_3^l\left(\partial_3E_\alpha^{\mathsf{e},app}-\partial_\alpha E_3^{\mathsf{e},app}\right)|_{x_3=0^+}+\varepsilon^2 f_\alpha,$$
$$E_j^{\mathsf{e},app}|_{x_3=\varepsilon} = E_j^{\mathsf{e},app}|_{x_3=0^+}+\varepsilon\partial_3E_j^{\mathsf{e},app}|_{x_3=0^+}+\varepsilon^2 g_j.$$

Moreover there exists a ε–independent constant $C>0$ such that

$$|f_\alpha|_{H^{1/2}(\Gamma)}\leqslant C, \; |g_j|_{H^{3/2}(\Gamma)}\leqslant C. \tag{51}$$

After simple calculations involving the explicit expressions of $(\mathcal{E}^{\mathsf{m},n})_{n=0,1,2}$ in local coordinates, transmission conditions (50) are written

$$\frac{1}{\mu_{\mathsf{e}}}\left(\partial_3W_\alpha^{\mathsf{e}}-\partial_\alpha W_3^{\mathsf{e}}\right)|_{x_3=\varepsilon^+} = \frac{1}{\mu_{\mathsf{m}}}\left(\partial_3W_\alpha^{\mathsf{m}}-\partial_\alpha W_3^{\mathsf{m}}\right)|_{x_3=\varepsilon^-}+\frac{\varepsilon^2}{\mu_{\mathsf{e}}}f_\alpha,$$
$$\frac{1}{\mu_{\mathsf{c}}}\left(\partial_3W_\alpha^{\mathsf{c}}-\partial_\alpha W_3^{\mathsf{c}}\right)|_{x_3=0^-} = \frac{1}{\mu_{\mathsf{m}}}\left(\partial_3W_\alpha^{\mathsf{m}}-\partial_\alpha W_3^{\mathsf{m}}\right)|_{x_3=0^+},$$
$$W_\alpha^{\mathsf{e}}|_{x_3=\varepsilon^+}=W_\alpha^{\mathsf{m}}|_{x_3=\varepsilon^-}+\varepsilon^2 g_\alpha, \text{ and } \quad W_\alpha^{\mathsf{c}}|_{x_3=0^-}=W_\alpha^{\mathsf{m}}|_{x_3=0^+}.$$

Observe that $\delta\mathbb{W}=-\dfrac{\varepsilon}{\mu_{\mathsf{m}}q_{\mathsf{m}}}1_{\mathcal{O}_{\mathsf{m}}^{\varepsilon}}\delta\mathbb{G}$, and the following equalities hold

$$q_{\mathsf{e}}W_3^{\mathsf{e}}|_{x_3=\varepsilon^+}=q_{\mathsf{m}}W_3^{\mathsf{m}}|_{x_3=\varepsilon^-}+q_{\mathsf{e}}\varepsilon^2 g_3, \quad q_{\mathsf{c}}W_3^{\mathsf{c}}|_{x_3=0^-}=q_{\mathsf{m}}W_3^{\mathsf{m}}|_{x_3=0^+}.$$

We choose $\mathbb{P}=p_i\mathrm{d}x^i$ in $H^2\Omega^1(\mathcal{O}_{\mathsf{e}}^{\varepsilon})$ such that

$$N_{\partial\mathcal{O}}\wedge\mathbb{P}|_{\partial\mathcal{O}}=0, \text{ and } \mathbb{P}|_{x_3=\varepsilon^+}=g_i(\mathsf{x}_{\mathsf{T}})\mathrm{d}x^i.$$

Since for $\varepsilon \in (0, d_0/2)$, the domain $\mathcal{O}_{\mathsf{e}}^{\varepsilon}$ satisfies $\mathcal{O}_{\mathsf{e}} \setminus (\mathscr{V}_\Gamma \cap \mathcal{O}_{\mathsf{e}}) \subset \mathcal{O}_{\mathsf{e}}^{\varepsilon} \subset \mathcal{O}_{\mathsf{e}}$, and according to (51), there exists an ε–independent constant $C > 0$ such that

$$\|\mathbb{P}\|_{H^2\Omega^1(\mathcal{O}_{\mathsf{e}}^{\varepsilon})} \leqslant C.$$

Defining $\widetilde{\mathbb{W}} = \mathbb{W} + \varepsilon^2 1_{\mathcal{O}_{\mathsf{e}}^{\varepsilon}}\mathbb{P}$, we infer

$$\delta\mathrm{d}\widetilde{\mathbb{W}} - \mu q\widetilde{\mathbb{W}} = \varepsilon^2 1_{\mathcal{O}_{\mathsf{e}}^{\varepsilon}} \left(\delta\mathrm{d}\mathbb{P} - \mu_{\mathsf{e}} q_{\mathsf{e}}\mathbb{P}\right) + \varepsilon 1_{\mathcal{O}_{\mathsf{m}}^{\varepsilon}}\mathbb{G}, \text{ in } \mathcal{O}, \quad N_{\partial\mathcal{O}} \wedge \widetilde{\mathbb{W}}|_{\partial\mathcal{O}} = 0, \quad \text{on } \partial\mathcal{O},$$

and the following transmission conditions hold

$$\begin{aligned}
&\frac{1}{\mu_{\mathsf{e}}}\left(\partial_3\widetilde{W}_\alpha^{\mathsf{e}} - \partial_\alpha\widetilde{W}_3^{\mathsf{e}}\right)|_{x_3=\varepsilon^+} = \frac{1}{\mu_{\mathsf{m}}}\left(\partial_3\widetilde{W}_\alpha^{\mathsf{m}} - \partial_\alpha\widetilde{W}_3^{\mathsf{m}}\right)|_{x_3=\varepsilon^-} + \frac{\varepsilon^2}{\mu_{\mathsf{e}}}\widetilde{f_\alpha},\\
&\frac{1}{\mu_{\mathsf{c}}}\left(\partial_3\widetilde{W}_\alpha^{\mathsf{c}} - \partial_\alpha\widetilde{W}_3^{\mathsf{c}}\right)|_{x_3=0^-} = \frac{1}{\mu_{\mathsf{m}}}\left(\partial_3\widetilde{W}_\alpha^{\mathsf{m}} - \partial_\alpha\widetilde{W}_3^{\mathsf{m}}\right)|_{x_3=0^+},\\
&\widetilde{W}_\alpha^{\mathsf{e}}|_{x_3=\varepsilon^+} = \widetilde{W}_\alpha^{\mathsf{m}}|_{x_3=\varepsilon^-}, \quad \widetilde{W}_\alpha^{\mathsf{c}}|_{x_3=0^-} = \widetilde{W}_\alpha^{\mathsf{m}}|_{x_3=0^+},
\end{aligned}$$

where $\widetilde{f_\alpha} = f_\alpha - \left(\partial_3 p_\alpha - \partial_\alpha p_3\right)|_{x_3=\varepsilon^+}$. Moreover

$$q_{\mathsf{e}}\widetilde{W}_3^{\mathsf{e}}|_{x_3=\varepsilon^+} = q_{\mathsf{m}}\widetilde{W}_3^{\mathsf{m}}|_{x_3=\varepsilon^-}, \quad q_{\mathsf{c}}\widetilde{W}_3^{\mathsf{c}}|_{x_3=0^-} = q_{\mathsf{m}}\widetilde{W}_3^{\mathsf{m}}|_{x_3=0^+}.$$

Since the functions $\widetilde{f_\alpha}$ are defined on Γ, it is convenient to define $\widetilde{F_\alpha}$ on Γ_ε by

$$\forall \mathbf{x}_{\mathsf{T}} \in \Gamma, \quad \widetilde{F_\alpha} \circ \Phi(\mathbf{x}_{\mathsf{T}}, \varepsilon) = \widetilde{f_\alpha}(\mathbf{x}_{\mathsf{T}}).$$

Denoting by $\widetilde{\mathbb{G}}$ and $\widetilde{\mathbb{F}}$ the following 1–forms defined by

$$\widetilde{\mathbb{G}} = \varepsilon 1_{\mathcal{O}_{\mathsf{m}}^{\varepsilon}}\mathbb{G} + \varepsilon^2 1_{\mathcal{O}_{\mathsf{e}}^{\varepsilon}}\left(\delta\mathrm{d}\mathbb{P} - \mu_{\mathsf{e}} q_{\mathsf{e}}\mathbb{P}\right), \quad \widetilde{\mathbb{F}} = \widetilde{F_\alpha}\mathrm{d}x^\alpha,$$

there exists an ε–independent constant $C > 0$ such that

$$\|\widetilde{\mathbb{G}}\|_{L^2\Omega^1(\mathcal{O})} \leqslant C\varepsilon^{3/2}, \quad \|\delta\widetilde{\mathbb{G}}\|_{L^2(\mathcal{O})} \leqslant C\varepsilon^{3/2} \text{ and } \quad \|\widetilde{\mathbb{F}}\|_{H^{-1/2}\Omega^1(\Gamma_\varepsilon)} \leqslant C.$$

The 1-form $\widetilde{\mathbb{W}}$ satisfies the following equalities

$$\delta\mathrm{d}\widetilde{\mathbb{W}} - \mu q\widetilde{\mathbb{W}} = \widetilde{\mathbb{G}}, \text{ in } \mathcal{O}_{\mathsf{e}}^{\varepsilon} \cup \mathcal{O}_{\mathsf{m}}^{\varepsilon} \cup \mathcal{O}_{\mathsf{c}}, \quad N_{\partial\mathcal{O}} \wedge \widetilde{\mathbb{W}}^{\mathsf{e}}|_{\partial\mathcal{O}} = 0, \quad \text{on } \partial\mathcal{O}, \tag{52a}$$

with the following transmission conditions on Γ_ε and on Γ

$$\frac{1}{\mu_{\mathsf{e}}}\mathsf{int}(N_{\Gamma_\varepsilon})\mathrm{d}\widetilde{\mathbb{W}}^{\mathsf{e}}|_{\Gamma_\varepsilon^+} = \frac{1}{\mu_{\mathsf{m}}}\mathsf{int}(N_{\Gamma_\varepsilon})\mathrm{d}\widetilde{\mathbb{W}}^{\mathsf{m}}|_{\Gamma_\varepsilon^-} + \frac{\varepsilon^2}{\mu_{\mathsf{e}}}\widetilde{\mathbb{F}}, \tag{52b}$$

$$\frac{1}{\mu_{\mathsf{m}}}\mathsf{int}(N_\Gamma)\mathrm{d}\widetilde{\mathbb{W}}^{\mathsf{m}}|_{\Gamma^+} = \frac{1}{\mu_{\mathsf{c}}}\mathsf{int}(N_\Gamma)\mathrm{d}\widetilde{\mathbb{W}}^{\mathsf{c}}|_{\Gamma^-}, \tag{52c}$$

$$N_{\Gamma_\varepsilon} \wedge \widetilde{\mathbb{W}}^{\mathsf{e}}|_{\Gamma_\varepsilon^+} = N_{\Gamma_\varepsilon} \wedge \widetilde{\mathbb{W}}^{\mathsf{m}}|_{\Gamma_\varepsilon^-}, \text{ and } \quad N_\Gamma \wedge \widetilde{\mathbb{W}}^{\mathsf{m}}|_{\Gamma^+} = N_\Gamma \wedge \widetilde{\mathbb{W}}^{\mathsf{c}}|_{\Gamma^-}. \tag{52d}$$

Moreover

$$\delta\widetilde{\mathbb{W}} = \frac{1}{\mu q}\delta\widetilde{\mathbb{G}}, \quad \text{in } \mathcal{O}_{\mathsf{e}}^{\varepsilon} \cup \mathcal{O}_{\mathsf{m}}^{\varepsilon} \cup \mathcal{O}_{\mathsf{c}}, \tag{53}$$

and $\widetilde{\mathbb{G}}$ and $\widetilde{\mathbb{F}}$ are such that

$$q_{\mathsf{e}}\mathsf{int}(N_{\Gamma_\varepsilon})\widetilde{\mathbb{W}}^{\mathsf{e}}|_{\Gamma_\varepsilon^+} = q_{\mathsf{m}}\mathsf{int}(N_{\Gamma_\varepsilon})\widetilde{\mathbb{W}}^{\mathsf{m}}|_{\Gamma_\varepsilon^-}, \quad q_{\mathsf{c}}\mathsf{int}(N_\Gamma)\widetilde{\mathbb{W}}^{\mathsf{c}}|_{\Gamma^-} = q_{\mathsf{m}}\mathsf{int}(N_\Gamma)\widetilde{\mathbb{W}}^{\mathsf{m}}|_{\Gamma^+}.$$

Multiply (52) by $\widetilde{\mathbb{W}}$ and integrate by parts with the help of (53) to infer

$$\|\widetilde{\mathbb{W}}^{\mathsf{e}}\|_{H\Omega^1(\mathrm{d},\delta,\mathcal{O}^\varepsilon_{\mathsf{e}})} + \sqrt{\varepsilon}\|\widetilde{\mathbb{W}}^{\mathsf{m}}\|_{H\Omega^1(\mathrm{d},\delta,\mathcal{O}^\varepsilon_{\mathsf{m}})} + \|\widetilde{\mathbb{W}}^{\mathsf{c}}\|_{H\Omega^1(\mathrm{d},\delta,\mathcal{O}_{\mathsf{c}})} \leqslant C\varepsilon^2,$$

for an ε–independent constant C. Morever $\widetilde{\mathbb{W}} = \mathbb{W} + \varepsilon^2 1_{\mathcal{O}^\varepsilon_{\mathsf{e}}}\mathbb{P}$ implies

$$\|\mathbb{W}^{\mathsf{e}}\|_{H\Omega^1(\mathrm{d},\delta,\mathcal{O}^\varepsilon_{\mathsf{e}})} + \sqrt{\varepsilon}\|\mathbb{W}^{\mathsf{m}}\|_{H\Omega^1(\mathrm{d},\delta,\mathcal{O}^\varepsilon_{\mathsf{m}})} + \|\mathbb{W}^{\mathsf{c}}\|_{H\Omega^1(\mathrm{d},\delta,\mathcal{O}_{\mathsf{c}})} \leqslant C\varepsilon^2, \tag{54}$$

from which we infer Theorem 2.8.

□

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Appendix 1
Asymptotic expansion at any order

We may extend our derivation principle to obtain asymptotic transmission conditions at any order. Actually, there exists a recurrence formula, which is given in this appendix. The sketch of the proof of the expansion, which is similar to the proof of Theorem 5.3 is left to the reader. For $(\alpha,\beta,\iota,\kappa) \in \{1,2\}^4$ define the following sequences $(A^l_{\alpha\beta\iota\kappa})_{l\in\mathbb{N}}$, $(B^l_{\alpha\beta\iota\kappa})_{l\in\mathbb{N}}$, $(C^l_{\alpha\beta\iota\kappa})_{l\in\mathbb{N}}$ and $(D^l_{\alpha\beta\iota\kappa})_{l\in\mathbb{N}}$ by

$$\begin{cases}
A^l_{\alpha\beta\iota\kappa} = \dfrac{\partial^l_\eta}{\varepsilon^l}\left(\dfrac{g^{\mathsf{m}}_{\alpha\iota}}{\sqrt{g^{\mathsf{m}}}}\dfrac{\partial_\eta}{\varepsilon}\left(\dfrac{g^{\mathsf{m}}_{\beta\kappa}}{\sqrt{g^{\mathsf{m}}}}\right)\right)\Bigg|_{\eta=0}, \\
B^l_{\alpha\beta\kappa} = \dfrac{\partial^l_\eta}{\varepsilon^l}\left(\dfrac{g^{\mathsf{m}}_{\alpha\beta}}{\sqrt{g^{\mathsf{m}}}}\partial_\kappa\left(\dfrac{1}{\sqrt{g^{\mathsf{m}}}}\right)\right)\Bigg|_{\eta=0}, \\
C^l_{\alpha\beta} = \dfrac{\partial^l_\eta}{\varepsilon^l}\left(\dfrac{g_{\alpha\beta}}{g^{\mathsf{m}}}\right)\Bigg|_{\eta=0}, \\
D^l = \dfrac{\partial^l_\eta}{\varepsilon^l}\left(\dfrac{1}{\sqrt{g^{\mathsf{m}}}}\dfrac{\partial_\eta}{\varepsilon}\left(\sqrt{g^{\mathsf{m}}}\right)\right)\Bigg|_{\eta=0}, \\
E^l_{\alpha\beta\kappa} = \dfrac{\partial^l_\eta}{\varepsilon^l}\left(\dfrac{1}{\sqrt{g^{\mathsf{m}}}}\partial_\alpha\left(\dfrac{g_{\beta\kappa}}{\sqrt{g^{\mathsf{m}}}}\right)\right)\Bigg|_{\eta=0}.
\end{cases}$$

Using (23)-(25), for $k \geq 1$ we define $\partial^2_\eta \mathcal{E}^{\mathsf{m},k+2}_\lambda$ and $\partial_\eta \mathcal{E}^{\mathsf{m},k+1}_3$ respectively by

$$\begin{aligned}
\partial^2_\eta \mathcal{E}^{\mathsf{m},k+2}_\lambda &= \partial_\eta\partial_\lambda \mathcal{E}^{\mathsf{m},k+1}_3 + \epsilon_{\alpha\beta 3}\epsilon_{\iota\kappa 3} A^0_{\lambda\alpha\iota\kappa}\partial_\eta \mathcal{E}^{\mathsf{m},k+1}_\beta - \mu_{\mathsf{m}} q_{\mathsf{m}} \mathcal{E}^{\mathsf{m},k}_\lambda \\
&+ \epsilon_{\alpha\beta 3}\epsilon_{\iota\kappa 3}\sum_{l=1}^{k}\Big\{\Big(B^l_{\lambda\iota\kappa}\partial_\alpha + C^l_{\lambda\iota}\partial_\kappa\partial_\alpha\Big)\mathcal{E}^{\mathsf{m},k-l}_\beta \\
&+ A^l_{\lambda\alpha\iota\kappa}\Big(\partial_\eta \mathcal{E}^{\mathsf{m},k+1-l}_\beta - \partial_\beta \mathcal{E}^{\mathsf{m},k-l}_3\Big)\Big\}, \\
\partial_\eta \mathcal{E}^{\mathsf{m},k+1}_3 &= -\sum_{l=0}^{k}\left(D^l \mathcal{E}^{\mathsf{m},k-l}_3 + \epsilon_{\alpha\beta 3}\epsilon_{\iota\kappa 3}\left(C^l_{\kappa\beta}\partial_\alpha + E^l_{\alpha\beta\kappa}\right)\mathcal{E}^{\mathsf{m},k-l}_\iota\right)
\end{aligned}$$

Define now the differential forms $\mathbb{S}_{k+1}$ and $\mathbb{T}_{k+1}$ by

$$\mathbb{S}_{k+1} = \Bigg\{ \frac{1}{\mu_{\mathsf{m}}} \int_0^1 \left(\partial_\eta^2 \mathcal{E}_\lambda^{\mathsf{m},k+2} - \partial_\lambda \partial_\eta \mathcal{E}_3^{\mathsf{m},k+1} \right) \mathrm{d}\eta - \frac{1}{\mu_{\mathsf{e}}} \sum_{l=0}^{k} \partial_{x_3}^l \left(\partial_3 \widetilde{\mathbb{E}}_\lambda^{\mathsf{e},k-l} - \partial_\lambda \widetilde{\mathbb{E}}_3^{\mathsf{e},k-l} \right) \Big|_{x_3=0^+} \Bigg\} \mathrm{d}x^\lambda,$$

$$\mathbb{T}_{k+1} = \left\{ \int_0^1 \partial_\eta \mathcal{E}_\lambda^{\mathsf{m},k+1} \mathrm{d}\eta - \sum_{l=0}^{k} \partial_{x_3}^l \widetilde{\mathbb{E}}_\lambda^{\mathsf{e},k-l} \right\} \mathrm{d}x^\lambda.$$

The 1–forms $\widetilde{\mathbb{E}}^{\mathsf{e},k+1}$ and $\mathbb{E}^{\mathsf{c},k+1}$ are therefore defined by

$$\begin{aligned} &\delta \mathrm{d} \widetilde{\mathbb{E}}^{\mathsf{e},k+1} - \mu_{\mathsf{e}} q_{\mathsf{e}} \widetilde{\mathbb{E}}^{\mathsf{e},k+1} = 0, \text{ in } \mathcal{O}_{\mathsf{e}}, \\ &\delta \mathrm{d} \mathbb{E}^{\mathsf{c},k+1} - \mu_{\mathsf{c}} q_{\mathsf{c}} \mathbb{E}^{\mathsf{c},k+1} = 0, \text{ in } \mathcal{O}_{\mathsf{c}}, \\ &N_{\partial\mathcal{O}} \wedge \widetilde{\mathbb{E}}^{\mathsf{e},k+1}|_{\partial\mathcal{O}} = 0, \end{aligned}$$

with the following transmission conditions on Γ

$$\begin{aligned} &\frac{1}{\mu_{\mathsf{e}}} \mathsf{int}(N_\Gamma) \mathrm{d} \widetilde{\mathbb{E}}^{\mathsf{e},k+1}|_{\Gamma^+} - \frac{1}{\mu_{\mathsf{c}}} \mathsf{int}(N_\Gamma) \mathrm{d} \mathbb{E}^{\mathsf{c},k+1}|_{\Gamma^-} = \mathbb{S}_{k+1}, \\ &N_\Gamma \wedge \widetilde{\mathbb{E}}^{\mathsf{e},k+1}|_{\Gamma^+} - N_\Gamma \wedge \mathbb{E}^{\mathsf{c},k+1}|_{\Gamma^-} = N_\Gamma \wedge \mathbb{T}_{k+1}. \end{aligned}$$

Since for $n = 0, 1$ the 1–forms $(\mathcal{E}^{\mathsf{m},n}, \mathbb{E}^{\mathsf{c},n}, \widetilde{\mathbb{E}}^{\mathsf{e},n})_{n=0,1}$ are determined by (10)–(48)–(46)–(49), and since $\partial_\eta \mathcal{E}_\lambda^{\mathsf{m},2}$ is also known according to Remark 4.2, the recurrence process is initialized. The reader could prove that outside a neighborhood of $\mathcal{O}_{\mathsf{m}}^\varepsilon$ the following estimate holds $\mathbb{E} = \sum_{k=0}^n \varepsilon^k \mathbb{E}^k + O(\varepsilon^n)$.

Appendix 2
Few notions of differential calculus

Basic notions on differential forms

In order to obtain a self-contained paper, we present here few notions of differential calculus. We refer the reader to the books of Schwarz [31], of Dubrovine *et al.* [9, 10] and of Flanders [15] for more precisions. In this paragraph, n is a positive integer and k is a non negative integer smaller than n.

Let $(M, \mathfrak{g})$ be a compact connected oriented Riemannian manifold of $\mathbb{R}^n$ with smooth compact boundary ∂M. For $p \in M$, T_pM denotes the tangent space to M at the point p. The tangent bundle TM is the disjoint union of the spaces T_pM, $p \in M$. We denote by $\Gamma(TM)$ the space of the smooth sections[9] of TM. We recall that the metric $\mathfrak{g}$ on the manifold M is a smooth map $\mathfrak{g} : TM \times TM \to \mathbb{R}$ such that for any $p \in M$, $\mathfrak{g}|_p : T_pM \times T_pM \to \mathbb{R}$ is bilinear, symmetric and positive definite.

Denote by $\Lambda^k(T_pM)$ the space of anti-symmetric k–linear maps and by $\Lambda^k(M)$ the exterior k-form bundle defined by $\Lambda^k(M) = \cup_{p\in M} \Lambda^k(T_pM)$. The

[9] There exists a smooth projection map π from the manifold TM unto M. A section of TM is a smooth map $s : M \to TM$ such that $\pi \circ s = \mathrm{Id}_M$.

space $\Omega^k(M)$ of k-forms is the space of all smooth sections of $\Lambda^k(M)$. We denote by $S(k,n)$ the set of the permutations σ (called (k,n)–shuffles) of the set $\{1,2,\cdots,n\}$ satisfying $\sigma(1) < \cdots < \sigma(k)$, $\sigma(k+1) < \cdots < \sigma(n)$, and by $\mathrm{sgn}(\sigma)$ the signature of the permutation $\sigma \in S(k,n)$. The following definitions and propositions come from Schwarz [31] and Flanders [15].

Definition-Proposition 2.1 (Elementary operations)**.** *Define the exterior, inner and interior products.*
• *The exterior product* $\wedge$ *of differential forms is defined by*

$$\wedge : \Omega^k(M) \times \Omega^l(M) \longrightarrow \Omega^{l+k}(M), \quad (\omega, \eta) \longmapsto \omega \wedge \eta,$$

where for arbitrary vector field $X = (X_1, \cdots, X_{k+l})$ *on* M*, we have*

$$\omega \wedge \eta(X) = \sum_{\sigma \in S(k,k+l)} \mathrm{sgn}(\sigma)\omega(X_{\sigma(1)}, \cdots, X_{\sigma(k)})\eta(X_{\sigma(k+1)}, \cdots, X_{\sigma(k+l)}).$$

• *Let* $(E_j)_{j=1}^n$ *be a local* $\mathfrak{g}$*–orthonormal*[10] *frame on* $U \subset M$*. The inner product on* $\Omega^k(M)$ *is locally defined from* $\Omega^k(M) \times \Omega^k(M)$ *to* $C^\infty(M)$ *by*

$$\langle \cdot, \cdot \rangle_{\Omega^k} : (\omega, \eta) \mapsto \langle \omega, \eta \rangle_{\Omega^k} = \sum_{\sigma \in S(k,n)} \omega(E_{\sigma(1)}, \cdots, E_{\sigma(k)})\eta(E_{\sigma(1)}, \cdots, E_{\sigma(k)}).$$

• *The Hodge star operator is defined by*

$$\star : \Omega^k(M) \longrightarrow \Omega^{n-k}(M), \quad \omega \mapsto \star\omega,$$

where $\star\omega$ *is the unique* $(n-k)$*–form satisfying*

$$\forall \eta \in \Omega^k, \quad \eta \wedge \star\omega = \langle \eta, \omega \rangle_{\Omega^k} \,\mathrm{dvol}_M\,.$$

The notation dvol_M *denotes the Riemannian volume* n*–form*

$$\mathrm{dvol}_M(X_1, \cdots, X_n) = \sqrt{det(\mathfrak{g}(X_i, X_j))}.$$

• *The interior product* $\mathsf{int}(Y)$ *with a smooth vector field* Y *of* $\Gamma(TM)$ *is defined by*

$$\mathsf{int}(Y) : \Omega^k(M) \longrightarrow \Omega^{k-1}(M), \quad \omega \mapsto \mathsf{int}(Y)\omega,$$

where

$$\Big(\mathsf{int}(Y)\omega\Big)(X_1, \cdots, X_{k-1}) = \omega(Y, X_1, \cdots, X_{k-1}), \, \forall (X_1, \cdots X_{k-1}) \in \Gamma(TM)^{k-1}.$$

Remark 2.2. [Identifications of spaces]The space $C^\infty(M)$ of smooth functions on M and $\Omega^0(M)$ coincide. Moreover by definition, $\Omega^1(M)$ is the cotangent bundle[11] T^*M. Therefore we may identify the space of vector fields $\Gamma(TM)$ with $\Omega^1(M)$. △

[10] For $U \subset M$, the tuple $(E_j)_{j=1}^n \in \Gamma(TU)^n$ is a local $\mathfrak{g}$–orthonormal frame on M if for any $p \in U$, $\mathfrak{g}(E_i, E_j)|_p = \delta_i^j$, where δ_i^j is the well-known Kronecker symbol equal to 1 if $i = j$ and 0 if $i \neq j$.
[11] The cotangent bundle T^*M is the disjoint union for $x \in M$ of linear forms on T_xM.

Remark 2.3. [Duality of the interior and exterior products] Denote by Y a vector field and identify Y with its corresponding 1–form. The interior product $\mathsf{int}(Y)$ is the dual map of the left exterior multiplication $Y\wedge$

$$\forall \omega \in \Omega^{k+1}(M),\, \forall \eta \in \Omega^k \quad \langle \mathsf{int}(Y)\omega, \eta\rangle_{\Omega^k(M)} = \langle \omega, Y \wedge \eta\rangle_{\Omega^{k+1}(M)}. \tag{2.01}$$

$\triangle$

Definition-Proposition 2.4 (Differential operators)**.** *We define the exterior differential, the codifferential and the Laplace-Beltrami operators.*
• *Let* $(\mathrm{d}y^j)_{j=1}^n$ *be a basis of* $\Omega^1(M)$*. There exists an unique differential operator* $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$*, called exterior differential, such that*

$$\begin{aligned}
&\forall (\omega, \eta) \in \Omega^k(M) \times \Omega^k(M), \quad \mathrm{d}(\omega + \eta) = \mathrm{d}\omega + \mathrm{d}\eta,\\
&\forall (\omega, \eta) \in \Omega^k(M) \times \Omega^l(M), \quad \mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^l \omega \wedge \mathrm{d}\eta,\\
&\forall \omega \in \Omega^k(M), \quad \mathrm{d}\,(\mathrm{d}\omega) = 0,\\
&\forall f \in C^\infty(M), \quad \mathrm{d}f : (y^1, \cdots, y^n) \in M \to \sum_{j=1}^n \frac{\partial f}{\partial y^j}(y^1, \cdots, y^n)\mathrm{d}y^j.
\end{aligned}$$

• *The codifferential is the map* $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ *defined by*

$$\forall \omega \in \Omega^k(M), \quad \delta\omega = (-1)^{nk+n+1} \star \mathrm{d}(\star\omega),\ \textit{if } k \neq 0 \textit{ and } \delta \sim 0 \textit{ on functions.}$$

• *The Laplace-Beltrami operator*[12] $\Delta : \Omega^k(M) \to \Omega^k(M)$ *is defined by*

$$\forall \omega \in \Omega^k(M), \quad \Delta\omega = -\left(\mathrm{d}\delta\omega + \delta\mathrm{d}\omega\right).$$

Appendix 2 is devoted to the explicit formulae of differential calculus in $\mathbb{R}^3$ equipped with the Euclidean metric in any system of coordinates. This formulae will be useful in the derivation of our asymptotic result.

Notation 2.5. *To simplify notations, and when it is evident, we omit the* $\wedge$ *symbol between 2 differential forms. For example, we denote by* $\mathrm{d}y^i\mathrm{d}y^j = \mathrm{d}y^i \wedge \mathrm{d}y^j$*. Observe that obviously* $\mathrm{d}y^i\mathrm{d}y^j = -\mathrm{d}y^j\mathrm{d}y^i$*.*

Define now the Sobolev spaces needed in the paper.

Definition 2.6 (Sobolev spaces)**.** *Let* $(U_a)_{a\in A}$ *be an open cover of* M*, let* $(\chi_a)_{a\in A}$ *be a subordinated partition of the unity and let* $(E_1^a, \cdots, E_n^a)$ *be a family of local frames. Denote by* ∇ *the induced Levi-Cività connection on* M*, and let* $\Omega_c^k(M)$ *be the space of the compactly supported* k*–forms on* M*. This space is equipped with the following* L^2*–inner product*

$$\langle\langle \omega, \eta\rangle\rangle = \int_M \langle \omega, \eta\rangle_{\Omega^k} \,\mathrm{dvol}_M\,.$$

For $s \in \mathbb{N}$*, define the* $H^s\Omega^k(M)$*–norm as follows*

$$\|\omega\|^2_{H^s\Omega^k(M)} = \sum_{a\in A} \int_M \chi_a |\omega|^2_{J^s(\Lambda)} \,\mathrm{dvol}_M,$$

[12] In order to identify Δ with the operator $\nabla \cdot (\nabla \cdot)$ we choose here to define the Laplace-Beltrami operator as the opposite of the geometric Laplacian.

where

$$|\omega|^2_{J^0(\Lambda)} = \langle \omega, \omega\rangle_{\Omega^k}, \quad |\omega|^2_{J^s(\Lambda)} = |\omega|^2_{J^{s-1}(\Lambda)} + \sum_{j=1}^{n} |\nabla_{E_j^a}\omega|^2_{J^{s-1}(\Lambda)}.$$

The Sobolev space $H^s\Omega^k(M)$ is defined as the completion of $\Omega^k_c(M)$ for the above $H^s\Omega^k(M)$–norm.

The Sobolev spaces $H^s\Omega^k(M)$, for $s \in \mathbb{R}$, are defined similary to the Sobolev spaces of functions: see Lions and Magenes [20].

Proposition 2.7 (Traces on ∂M for $H\Omega^k(\mathrm{d}, M)$ and $H\Omega^k(\delta, M)$)**.** *Denote by $\mathcal{J}$ the natural embedding $\partial M \to M$ and $\mathcal{J}^*$ its pull-back $\mathcal{J}^* : \Omega^k(M) \to \Omega^k(\partial M)$. Denote by $N_{\partial M}$ the outward normal*[13] *to ∂M. The following traces hold [31, 18, 24]*

$$\text{for all } \omega \in H\Omega^k(\mathrm{d}, M), \quad \mathcal{J}^*(\omega) \in H^{-1/2}\Omega^k(\partial M), \tag{2.02}$$

$$\text{for all } \omega \in H\Omega^k(\delta, M), \quad \mathrm{int}(N_{\partial M})\omega \in H^{-1/2}\Omega^k(\partial M), \tag{2.03}$$

$$\text{for all } s \geq 0, \text{ for all } \omega \in H^s\Omega^k(M), \quad \omega|_{\partial M} \in H^{s-1/2}\,\Omega^k(M)\big|_{\partial M}. \tag{2.04}$$

Moreover the usual Sobolev embeddings hold true for the k–forms. The following Green formula is useful (see [31, 18, 24]).

Proposition 2.8 (Green formula)**.** *Denote by N the outward normal to ∂M. Let $(\omega, \eta) \in H\Omega^{k-1}(\mathrm{d}, M) \times H\Omega^k(\delta, M)$, such that either $\mathcal{J}^*\omega$ or $\mathrm{int}(N)\eta$ belongs to $H^{1/2}\Omega^{k-1}(\partial M)$. The following equality holds*

$$\int_M \langle \mathrm{d}\omega, \eta\rangle_{\Omega^k}\,\mathrm{dvol}_M = \int_M \langle \omega, \delta\eta\rangle_{\Omega^{k-1}}\,\mathrm{dvol}_M - \int_{\partial M} \langle \omega, \mathsf{int}(N)\eta\rangle_{\Omega^{k-1}}\mathrm{d}\sigma_{\partial M}.$$

We denote by $\mathrm{d}\sigma_{\partial M}$ the surface form of ∂M, in order to differentiate the volume form of M and the surface form of ∂M.

Property 2.9 (Useful equality)**.** *Suppose now that M is a compact connected oriented Riemanian manifold without boundary. Let ω is a k–form and Y is a smooth 1–form such that $\mathrm{d}Y = 0$. Then applying the above Green formula with the help of equality (2.01) we infer that for $\omega \in H\Omega^k(\delta, M)$*

$$\mathsf{int}(Y)\delta\omega = (-1)^k\delta\left(\mathsf{int}(Y)\omega\right). \tag{2.05}$$

[13]Denote by $\Gamma(TM|_{\partial M})$ the space of vector fields on M sitting over the boundary ∂M. A field $N \in \Gamma(TM|_{\partial M})$ is a unit normal field on M is $\mathfrak{g}(N, N) = 1$ and for any $Y \in \Gamma(T\partial M)$, $\mathfrak{g}(Y, N) = 0$. Therefore observe that $\mathrm{d}N = 0$.

Proof. Actually, for any $\eta \in H\Omega^{k-2}(\mathrm{d}, M)$, we have

$$\begin{aligned}\int_M \langle \mathsf{int}(Y)\delta\omega, \eta\rangle_{\Omega^{k-2}} \,\mathrm{dvol}_M &= \int_M \langle \delta\omega, Y \wedge \eta\rangle_{\Omega^{k-1}} \,\mathrm{dvol}_M \\ &= \int_M \langle \omega, \mathrm{d}(Y \wedge \eta)\rangle_{\Omega^k} \,\mathrm{dvol}_M, \\ &= (-1)^{k-2} \int_M \langle \omega, Y \wedge \mathrm{d}\eta\rangle_{\Omega^k} \,\mathrm{dvol}_M \\ &= (-1)^{k-2} \int_M \langle \mathsf{int}(Y)\omega, \mathrm{d}\eta\rangle_{\Omega^{k-1}} \,\mathrm{dvol}_M \\ &= (-1)^{k-2} \int_M \langle \delta\left(\mathsf{int}(Y)\omega\right), \eta\rangle_{\Omega^{k-2}} \,\mathrm{dvol}_M \,.\end{aligned}$$

□

Explicit formulae

The definitions and propositions of the previous paragraph recall the basic notions of differential calculus for a general compact connected oriented Riemannian manifold $(M, \mathfrak{g})$ of $\mathbb{R}^n$ with smooth compact boundary ∂M. Present now the explicit formulae of the differential calculus for a manifold $M \subset \mathbb{R}^3$ equipped with the Euclidean metric. Denote by (x, y, z) the usual Euclidean coordinates of M and let (y_1, y_2, y_3) another system of coordinates: there exists a C^∞–diffeomrphism ψ such that $\psi(y_1, y_2, y_3) = (x, y, z)$. The Euclidean metric in (y_1, y_2, y_3)–coordinates is given by the matrix $(g_{ij})_{i,j=1,2,3}$: $\quad g_{ij} = \partial_{y_i}\psi \cdot \partial_{y_j}\psi$, where $\cdot$ denotes the Euclidean scalar product of $\mathbb{R}^3$. The inverse matrix of $(g_{ij})_{ij}$ is denoted by $(g^{ij})_{ij}$ and set

$$g = \det\Big((g_{ij})_{i,j=1,2,3}\Big).$$

Denote by $(\mathrm{d}y^1, \mathrm{d}y^2, \mathrm{d}y^3)$ the basis of $\Omega^1(M)$ associated to (y_1, y_2, y_3). It is clear that 2–forms $(\mathrm{d}y^2 \wedge \mathrm{d}y^3, \mathrm{d}y^3 \wedge \mathrm{d}y^1, \mathrm{d}y^1 \wedge \mathrm{d}y^2)$ is a basis of $\Omega^2(M)$. Since M is equipped with the Euclidean metric, we perform the change of coordinates $\psi(y_1, y_2, y_3) = (x, y, z)$ to infer that the inner product $\langle ., .\rangle_{\Omega^k}$ for $k = 0, 1, 2$, is determined in (y_1, y_2, y_3)–coordinates by[14] the following equalities

$$\langle F, G\rangle_{\Omega^0} = FG, \tag{2.06a}$$

$$\langle \mathrm{d}y^i, \mathrm{d}y^j\rangle_{\Omega^1} = g^{ij}, \tag{2.06b}$$

$$\langle \mathrm{d}y^i \mathrm{d}y^k, \mathrm{d}y^j \mathrm{d}y^l\rangle_{\Omega^2} = g^{ij}g^{kl} - g^{il}g^{jk}, \tag{2.06c}$$

$$\langle F\mathrm{d}y^1\mathrm{d}y^2\mathrm{d}y^3, G\mathrm{d}y^1\mathrm{d}y^2\mathrm{d}y^3\rangle_{\Omega^3} = \frac{1}{g}FG, \tag{2.06d}$$

where F and G are smooth functions on M, and g is the determinant of (g_{ij}).

• **Exterior products on $\mathbb{R}^3$.** The exterior product between a k–form and a l–form equals zero as soon a $k + l > 3$. Moreover, for $k \in \{0, \cdots, 3\}$, the exterior product between a 0–form and a k–form is the usual scalar multiplication between a function and a k–form. Accordingly Definition-Proposition 2.1, the

[14]To simplify notations, we omit the sign $\wedge$ between the differential forms $\mathrm{d}y^i$ and $\mathrm{d}y^j$, for $i, j = 1, 2, 3$.

following formulae hold.
▷ Exterior product of 1–forms. Let $\lambda = \lambda_i \mathrm{d}y^i$ and $\mu = \mu_i \mathrm{d}y^i$ be two 1-forms, then

$$\lambda \wedge \mu = \lambda_i \mu_j \mathrm{d}y^i \mathrm{d}y^j = \frac{\epsilon_{ijk}}{2} \left(\epsilon_{klm} \lambda_l \mu_m\right) \mathrm{d}y^i \mathrm{d}y^j.$$

▷ Exterior product between a 2–form and a 1–form. Let $\lambda = \frac{\epsilon_{ijk}}{2} \lambda_k \mathrm{d}y^i \mathrm{d}y^j$ and $\mu = \mu_i \mathrm{d}y^i$, then

$$\lambda \wedge \mu = \lambda_k \mu_k \mathrm{d}y^1 \mathrm{d}y^2 \mathrm{d}y^3.$$

• **Expression of** d**.** A straigthforward application of the reccurence formula given in Definition-Proposition 2.4 implies the following formulae.

▷ d *on 0–forms.* Let λ be a 0–form, *i.e.* λ is a function. Then

$$\mathrm{d}\lambda = \frac{\partial \lambda}{\partial y_i} \mathrm{d}y^i.$$

▷ d *on 1–forms.* Let $\mu = \mu_i \mathrm{d}y^i$, then $\mathrm{d}\mu$ equals

$$\mathrm{d}\mu = \frac{\partial \mu_j}{\partial y_i} \mathrm{d}y^i \mathrm{d}y^j = \frac{\epsilon_{ijk}}{2} \left(\epsilon_{klm} \frac{\partial \mu_m}{\partial y_l} \right) \mathrm{d}y^i \mathrm{d}y^j.$$

▷ d *on 2–forms.* Let $\lambda = \frac{\epsilon_{ijk}}{2} \lambda_k \mathrm{d}y^i \mathrm{d}^j$ be a 2–form, then we have

$$\mathrm{d}\lambda = \frac{\partial \lambda_k}{\partial y_k} \mathrm{d}y^1 \mathrm{d}y^2 \mathrm{d}y^3.$$

Proposition 2.10 (Star Hodge operator)**.** *Star Hodge operator is defined by Definition-Proposition 2.1.*
• **Hodge on functions and 3–forms.** *Let S be a 0-form and $T = \tau \, \mathrm{d}y^1 \mathrm{d}y^2 \mathrm{d}y^3$ be a 3-form. Then*

$$\star S = \sqrt{g} S \, \mathrm{d}y^1 \mathrm{d}y^2 \mathrm{d}y^3, \quad \star T = \frac{1}{\sqrt{g}} \tau.$$

• **Hodge on 1–forms.** *Let $R = R_i \, \mathrm{d}y^i$ be a 1-form. Then $\star R$ is the 2-form defined by*

$$\star R = \frac{\epsilon_{ijk}}{2} \sqrt{g} g^{kl} R_l \, \mathrm{d}y^i \mathrm{d}y^j.$$

• **Hodge on 2–forms.** *Let $S = \frac{\epsilon_{ijk}}{2} S_k \, \mathrm{d}y^i \mathrm{d}y^j$ be a 2-form. Then $\star S$ is the 1-form equal to*

$$\star S = \frac{1}{\sqrt{g}} g_{ik} S_k \, \mathrm{d}y^i.$$

Proof. According to Definition-Proposition 2.1, if ω is a k–form, then $\star\omega$ is the $3 - k$ form such that

$$\forall \eta \in \Omega^k(M), \quad \eta \wedge \star\omega = \langle \eta, \omega \rangle_{\Omega^k(M)} \sqrt{g} \, \mathrm{d}y^1 \mathrm{d}y^2 \mathrm{d}y^3.$$

Applying the above formulae of the exterior products, and equalities (2.06), we infer the proposition. □

Proposition 2.11 (The codifferential operator δ)**.** *The codifferential is defined by Definition-Proposition 2.4.*
• **Codifferential of 1-forms.** *Let* $\mu = \mu_i \mathrm{d}y^i$*, then*

$$\delta\mu = -\frac{1}{\sqrt{g}}\frac{\partial}{\partial y_k}\left(\sqrt{g}g^{kl}\mu_l\right). \tag{2.07}$$

• **Codifferential of 2–forms.** *Let* $\lambda = \dfrac{\epsilon_{ijk}}{2}\lambda_k \mathrm{d}y^i \mathrm{d}y^j$*, then*

$$\delta\lambda = \epsilon_{jkl}\frac{g_{ij}}{\sqrt{g}}\frac{\partial}{\partial y_k}\left(\frac{g_{lm}}{\sqrt{g}}\lambda_m\right)\mathrm{d}y^i.$$

Proof. Since the codifferential on k–forms in $\mathbb{R}^3$ is defined by $\delta = (-1)^{3k} \star \mathrm{d}\star$, a straightforward application of the formulae of the differential operator d and the use of Proposition 2.10 lead to the formulae of the codifferential operator. □

Proposition 2.11 with the formulae of d differential operator implies the following corollary.

Corollary 2.12 (δd and Δ operators on functions and on 1–forms)**.** *Recall that* $\Delta = -\left(\delta\mathrm{d} + \mathrm{d}\delta\right)$*.*
• *Let f be a function. Then*

$$\Delta f = -\delta\mathrm{d}f = \frac{1}{\sqrt{g}}\frac{\partial}{\partial y_k}\left(\sqrt{g}g^{kl}\frac{\partial}{\partial y_l}f\right). \tag{2.08}$$

• *Let* $\lambda = \lambda_i \mathrm{d}y^i$ *be a 1–form, then*

$$\delta\mathrm{d}\lambda = \epsilon_{ijk}\epsilon_{lmn}\frac{g_{ri}}{\sqrt{g}}\frac{\partial}{\partial y_j}\left(\frac{g_{kl}}{\sqrt{g}}\frac{\partial}{\partial y_m}\lambda_n\right)\mathrm{d}y^r, \tag{2.09}$$

$$\Delta\lambda = -\left(\epsilon_{ijk}\epsilon_{lmn}\frac{g_{ri}}{\sqrt{g}}\frac{\partial}{\partial y_j}\left(\frac{g_{kl}}{\sqrt{g}}\frac{\partial}{\partial y_m}\lambda_n\right) - \frac{\partial}{\partial y_r}\left(\frac{1}{\sqrt{g}}\frac{\partial}{\partial y_k}\left(\sqrt{g}g^{kl}\lambda_l\right)\right)\right)\mathrm{d}y^r. \tag{2.010}$$

Using equality (2.01), we infer the following proposition.

Proposition 2.13 (Interior product)**.** *Let N be a vector-field identified with the corresponding 1–form* $N = N_i \mathrm{d}y^i$*.*
• **Interior product of a vector-field on a 1–form.** *Let* $\mu = \mu_i \mathrm{d}y^i$*. Then*

$$\mathsf{int}(N)\mu = g^{ij}N_j\mu_i. \tag{2.011}$$

• **Interior product of a vector-field on a 2–form.** *Let* $\mu = \mu_{ij}\,\mathrm{d}y^i \mathrm{d}y^j$*, then*

$$\mathsf{int}(N)\mu = g_{rl}\mu_{ij}N_k\left(g^{ik}g^{jl} - g^{il}g^{jk}\right)\,\mathrm{d}y^r. \tag{2.012}$$

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